

CONNES-KREIMER QUANTIZATIONS AND PBW THEOREMS FOR PRE-LIE ALGEBRAS

by

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Abstract. — The Connes-Kreimer renormalization Hopf algebras are examples of a canonical quantization procedure for pre-Lie algebras. We give a simple construction of this quantization using the universal enveloping algebra for so-called twisted Lie algebras (Lie algebras in the category of symmetric sequences of \mathbf{k} -modules). As an application, we obtain a simple proof of the (quantized) PBW theorem for Lie algebras which come from a pre-Lie product (over an arbitrary commutative ring). More generally, we observe that the quantization and the PBW theorem extend to pre-Lie algebras in arbitrary abelian symmetric monoidal categories with limits. We also extend a PBW theorem of Stover for connected twisted Lie algebras to this categorical setting.

Résumé. — Les algèbres de Hopf de Connes-Kreimer, utilisées en renormalisation, sont des exemples de procédés de quantification canoniques pour les algèbres pré-Lie. On donne une construction simple de cette quantification en utilisant l'algèbre enveloppante universelle des “algèbres de Lie tordues” (algèbres de Lie dans la catégorie des modules symétriques). Comme application on obtient une démonstration simple du théorème PBW (quantifié) pour les algèbres de Lie issues d'un produit pré-Lie (sur un anneau de base commutatif quelconque). Plus généralement, on observe que la quantification et le théorème de PBW s'étendent aux algèbres pré-Lie dans n'importe quelle catégorie symétrique monoidale abélienne avec limites. On étend aussi un théorème de Stover pour les algèbres de Lie tordues connexes dans ce contexte catégorique.

Contents

1. Introduction.....	1
2. Theorem 1.1.4 using the graded PBW theorem.....	6
3. Pre-Lie algebras as twisted Lie algebras.....	7
4. Generalizations and full proof of Theorem 1.1.4.....	11
5. Graded PBW theorems in a unified categorical context.....	20
A. PBW counterexamples and pre-Lie identities.....	22
References.....	23

1. Introduction

1.1. Connes-Kreimer renormalization algebras. — Connes and Kreimer introduced [CK98] a renormalization Hopf algebra to organize computations involved in certain Feynman diagram expansions. The dual of this Hopf algebra is given by $(\mathrm{Sym} \, \mathfrak{g}, \Delta, *)$ where \mathfrak{g} is

2000 Mathematics Subject Classification. — 17D99, 17B35.

Key words and phrases. — pre-Lie algebras, PBW theorems, renormalization, Hopf algebras, twisted Lie algebras, S-modules.

the vector space (or \mathbf{k} -module for \mathbf{k} an arbitrary commutative ring) freely generated by rooted trees, Δ is the standard coproduct on $\text{Sym } \mathfrak{g}$, and $*$ satisfies, for $x \in \text{Sym}^a \mathfrak{g}, y \in \text{Sym}^b \mathfrak{g}$,

$$(1.1.1) \quad x * y = xy + \text{terms in degrees } a, a+1, \dots, a+b-1.$$

In particular, if x and y are forests of rooted trees (i.e., monomials in $\text{Sym } \mathfrak{g}$), then $x * y$ is the sum of all ways of grafting the trees of y to distinct branches of x (or simply adding the trees to the forest without grafting). We will not be concerned further with the specific formula.

Connes and Kreimer observed in [CK98] that $(\text{Sym } \mathfrak{g}, \Delta, *) \cong U\mathfrak{g}$, the universal enveloping algebra of \mathfrak{g} equipped with the bracket $\{x, y\} = x * y - y * x$, as filtered Hopf algebras. Chapoton and Livernet further noted in [CL01] that \mathfrak{g} is not just a Lie algebra but a (right) *pre-Lie algebra* (and, in fact, a free pre-Lie algebra). Pre-Lie algebras generalize associative algebras; as defined in [Ger63, Vin63], they consist of a multiplication \circ on \mathfrak{g} satisfying

$$(1.1.2) \quad x \circ (y \circ z) - (x \circ y) \circ z = x \circ (z \circ y) - (x \circ z) \circ y.$$

As in the associative case, every pre-Lie algebra (\mathfrak{g}, \circ) as above has an associated Lie bracket,

$$(1.1.3) \quad \{x, y\} := x \circ y - y \circ x.$$

In [OG05, OG08], Oudom and Guin produced from an arbitrary pre-Lie algebra an explicit, interesting multiplication $*$ such that $(\text{Sym } \mathfrak{g}, \Delta, *) \cong U\mathfrak{g}$, and in [GS08], this construction was used to prove the following theorem (stated dually in [GS08, Proposition 3.5.2]):

Theorem 1.1.4. — [GS08] *Let \mathbf{k} be any commutative ring. Star products $*$ on $\text{Sym } \mathfrak{g}$ satisfying the conditions*

- (i) *$*$ forms a bialgebra with the usual coproduct Δ ,*
- (ii) *$*$ is a filtered product whose associated graded is the usual product on $\text{Sym } \mathfrak{g}$,*
- (iii) *$*$ satisfies*

$$(1.1.5) \quad (\text{Sym}^m \mathfrak{g}) * \text{Sym } \mathfrak{g} \subseteq (\text{Sym}^m \mathfrak{g})(\text{Sym } \mathfrak{g}) = \text{Sym}^{\geq m} \mathfrak{g},$$

are equivalent to (right) pre-Lie algebra structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, under the correspondence

$$(1.1.6) \quad x * y = xy + x \circ y, \quad \forall x, y \in \mathfrak{g}.$$

Under this equivalence, the dual to the renormalization Hopf algebra, on the nose, is obtained from the much simpler pre-Lie algebra \mathfrak{g} spanned by rooted trees. (Although this Hopf algebra is, by [CK98], canonically isomorphic to $U\mathfrak{g}$, the precise star product and the isomorphism with $U\mathfrak{g}$ are not defined merely by the Lie bracket on \mathfrak{g} , but require the pre-Lie structure).

1.2. Star product formulas. — The difficult part of the proof of the theorem is the construction of $*$ from an arbitrary pre-Lie algebra, which was done in [OG05, OG08] using complicated explicit computations, based on the following formulas related to rooted trees:

- (0) $a \circ 1 = a$,
- (1) $a \circ (bx) = (a \circ b) \circ x - a \circ (b \circ x), \quad \forall a, b \in \text{Sym } \mathfrak{g}, x \in \mathfrak{g}$,
- (2) $(ab) \circ c = (a \circ c')(b \circ c'')$,
- (3) $a * b = (a \circ b')b''$,

To deduce a general formula for $*$, (0), (1), and (2) first extend \circ to a certain binary operation on all of $\text{Sym } \mathfrak{g}$ (which is *not* a pre-Lie multiplication) by induction on degree, and then (3) expresses $*$ using this. In parts (2) and (3), we use Sweedler notation $\Delta(a) = a' \otimes a''$, which is shorthand for $\sum_i a'_i \otimes a''_i$, for some $a'_i, a''_i \in \text{Sym } \mathfrak{g}$.

The original goal of this paper was to give an alternative, more conceptual proof of Theorem 1.1.4, avoiding complicated calculations with the above formulas. We succeed in this (in §2) assuming that the graded PBW theorem, $\text{Sym } \mathfrak{g} \cong \text{gr } U\mathfrak{g}$, holds for $(\mathfrak{g}, \{, \})$. In full generality, we later prove the theorem as a consequence of the stronger Theorem 4.3.1.

The above formulas (0)–(3) follow immediately from Theorem 1.1.4, as we explain now:

Notation 1.2.1. — Let $\pi_n : \text{Sym } \mathfrak{g} \rightarrow \text{Sym}^n \mathfrak{g}$ be the projection to degree n .

Proposition 1.2.2. — Let \mathfrak{g} be any \mathbf{k} -module. Given any star product $*$ on $\text{Sym } \mathfrak{g}$ satisfying (i), (ii), and (iii) from Theorem 1.1.4, define $\circ : \text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g}$ by $g \circ h := \pi_n(g * h)$ for all $g \in \text{Sym}^n \mathfrak{g}$, $h \in \text{Sym } \mathfrak{g}$, extended linearly. Then, formulas (0)–(3) above hold.

The reader not interested in the following proof can safely skip it.

Proof. — (0) Since $a * 1 = a$, this follows immediately.

(1) If $a \in \text{Sym}^n \mathfrak{g}$, then $a \circ (bx) = \pi_n(a * (b * x - b \circ x)) = \pi_n((a * b) * x - a * (b \circ x)) = (a \circ b) \circ x - a \circ (b \circ x)$.

(2),(3) We can combine these into the single formula

$$(1.2.3) \quad (ab) * c = (a \circ c')(b \circ c'')c''''.$$

This identity is obvious in the case that $c \in \mathfrak{g}$, so inductively assume it holds for $c \in \text{Sym}^{\leq n} \mathfrak{g}$. The inductive step follows since, for c and d of degrees between 1 and n ,

$$\begin{aligned} (ab) * (c * d) &= ((ab) * c) * d = ((a \circ c')(b \circ c'')c''') * d \\ &= ((a \circ c') \circ d')((b \circ c'') \circ d'')(c''' \circ d''')d'''' = (a \circ (c' * d'))(b \circ (c'' * d''))(c''' * d'''). \quad \square \end{aligned}$$

1.3. PBW theorems. — Theorem 1.1.4 has the following interesting corollary (which the author did not find mentioned in the literature):

Corollary 1.3.1. — (i) (*Pre-Lie graded PBW theorem*) If \mathfrak{g} is the associated Lie algebra of a pre-Lie algebra over an arbitrary commutative ring \mathbf{k} , then $U\mathfrak{g}$ is a filtered Hopf algebra such that $\text{Sym } \mathfrak{g} \xrightarrow{\sim} \text{gr } U\mathfrak{g}$ by the canonical map.

(ii) (*Pre-Lie quantum PBW theorem*)⁽¹⁾ The map lifts to a coalgebra isomorphism $\text{Sym } \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$.

The pre-Lie assumption has a different flavor from the assumptions of the classical PBW theorems, which impose conditions on the \mathbf{k} -module structure of \mathfrak{g} rather than on its Lie structure. In particular, if \mathfrak{g} is the associated Lie algebra of an arbitrary associative algebra, then the PBW theorem holds for \mathfrak{g} , regardless of its \mathbf{k} -module structure. This result already seems hard to find in the literature.⁽²⁾

Note that, in our one-page proof in §2 of Theorem 1.1.4, we actually assume the graded PBW theorem, which is part (i) of the corollary above, or alternatively work under an assumption on the \mathbf{k} -module structure of \mathfrak{g} that implies this, as below. However, we later give a proof which avoids such an assumption, using Theorem 4.3.1, which relies instead on Stover’s graded

⁽¹⁾We use here the terminology “quantum PBW” since part (ii) says in particular that $U\mathfrak{g}$ is a filtered quantization of $\text{Sym } \mathfrak{g}$ (i.e., $U\mathfrak{g}$ is an associative algebra such that $\text{gr } U\mathfrak{g} \cong \text{Sym } \mathfrak{g}$ as Poisson algebras).

⁽²⁾Note that an associative algebra can be viewed as either a right or a left pre-Lie algebra, and the latter corresponds to using the opposite multiplication of $*$, which gives a different explicit quantum PBW isomorphism $\text{Sym } \mathfrak{g} \rightarrow U\mathfrak{g}$. This seems to indicate that this method of proving quantum PBW is not entirely natural for associative algebras. It is tempting to look for an abstract proof, especially since the result extends to associative algebras in general categorical settings (§4.7), but we couldn’t find it.

PBW theorem [Sto93] in the category of \mathbb{S} -modules that imposes no condition on the \mathbf{k} -module structure. This is one motivation for the material on \mathbb{S} -modules that form the heart of this work: it gives a route (different from [OG05, OG08]) to prove the above PBW theorems even in case of \mathbf{k} -modules for which (i) does not necessarily hold, or is not known to hold.

The standard contexts in which the PBW theorem is known include (cf. [Hig69]):

- (1) \mathfrak{g} is a free \mathbf{k} -module [Poi00, Bir37, Wit37] (or, more generally, a direct sum of cyclic modules);
- (2) $\mathbf{k} \supseteq \mathbb{Q}$ [Coh63].

Moreover, in these cases, the quantum PBW theorem holds, using an explicit lift $\text{Sym } \mathfrak{g} \cong U\mathfrak{g}$ of the graded PBW isomorphism: in case (1), one may obtain a PBW coalgebra isomorphism $\text{Sym } \mathfrak{g} \cong U\mathfrak{g}$ using explicit bases for free (or cyclic) \mathbf{k} -modules, and in case (2) one has the symmetrization map $\text{Sym } \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$ of coalgebras, sending $x_1 \cdots x_n$ to $\frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$. (See §5 for sketches of proofs.) These approaches do not seem to apply to the case of general pre-Lie algebras over an arbitrary commutative ring. However, in these contexts, the proof of §2 of Theorem 1.1.4 suffices (and one obtains a generally different lift of the isomorphism $\text{Sym } \mathfrak{g} \cong \text{gr } U\mathfrak{g}$ to a coalgebra isomorphism $\text{Sym } \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$ than the above).

Remark 1.3.2. — There are many other cases where at least the graded PBW theorem holds, although it is no longer clear whether the quantum PBW theorem holds. For example, because the graded PBW theorem holds when $\mathbf{k} \supseteq \mathbb{Q}$, it must hold more generally when \mathfrak{g} is torsion-free over \mathbb{Z} , because one can tensor with \mathbb{Q} . As another example, if all localizations of \mathfrak{g} at prime ideals of \mathbf{k} are direct sums of cyclic modules (or torsion-free over \mathbb{Z}), or direct limits thereof, then the graded PBW theorem must hold. For instance, this happens whenever \mathbf{k} is a Dedekind domain (which is a third classical case where the graded PBW theorem holds, attributed to [Laz54, Car58]), or whenever \mathfrak{g} is a flat \mathbf{k} -module.

Remark 1.3.3. — In the appendix, we recall an example from [Coh63] where the graded PBW isomorphism fails (and \mathbf{k} is an \mathbb{F}_p -algebra). Such examples for $p = 2$ are even older: see, e.g., [Šir53, Car58].

1.4. Categorical generalization. — In §4.7 below, we observe that the above construction makes sense in an *arbitrary* symmetric monoidal category (which is abelian with arbitrary limits), and the star-product and (quantum) PBW theorems therefore hold in this generality. In particular, the associated Lie algebra of any pre-Lie algebra (or associative algebra) object satisfies the PBW theorem.

Moreover, to prove this theorem, we prove a categorical generalization of Stover’s graded PBW theorem, replacing \mathbf{k} -modules by an arbitrary symmetric monoidal category as above. Hence, connected twisted Lie algebras are replaced by Lie algebras in the category of symmetric sequences of objects of such a category.

Although we defer the precise explanations and definitions to that section, it is worth pointing out a simple case where this is nontrivial:

Example 1.4.1. — If we work in the category of $\mathbb{Z}/2$ -graded modules over \mathbf{k} equipped with the super braiding $(x \otimes y \mapsto (-1)^{|x||y|} y \otimes x)$, the resulting Lie algebras are commonly called Lie superalgebras. When such a Lie superalgebra \mathfrak{g} is a free \mathbf{k} -module, then the graded PBW isomorphism $\text{Sym } \mathfrak{g} \xrightarrow{\sim} \text{gr } U\mathfrak{g}$ holds if and only if $\{x, x\} = 0$ for all even $x \in \mathfrak{g}$ and $\{x, \{x, x\}\} = 0$ for all odd $x \in \mathfrak{g}$ (cf. Remark 5.0.12). Note that these conditions do *not* hold for all Lie superalgebras, and the second condition is not even true for all Lie superalgebras for which

$\{x, y\}$ is the antisymmetrization of a binary operation. However, in a pre-Lie superalgebra, the pre-Lie axiom implies that $2x \circ (x \circ x) = 2(x \circ x) \circ x$ for all odd $x \in \mathfrak{g}$, and hence $\{x, \{x, x\}\} = 0$.

1.5. Twisted algebras. — To prove Theorem 1.1.4 without assumptions on \mathfrak{g} and \mathbf{k} such as (1) or (2) of §1.3, we exploit a connection between pre-Lie algebras and twisted Lie algebras, which should be interesting in its own right. Here, a twisted Lie algebra (in the sense of, e.g., [Bar78, Joy86])⁽³⁾ is defined as a Lie algebra object in a certain category which replaces that of \mathbf{k} -modules. The category is that of \mathbb{S} -modules (otherwise known as symmetric sequences of \mathbf{k} -modules, or species). This category is well known: for example, \mathbf{k} -linear operads are a different type of monoidal object in this category, and similar categories are used to define various types of spectra in topology. We recall more precisely the definition of this category in §3.2 below, and speak informally in this section for the benefit of the reader not familiar with these notions.

Our main observation is that pre-Lie algebra structures on a \mathbf{k} -module \mathfrak{g} are equivalent to twisted Lie algebra structures on the suspension $\Sigma\mathfrak{g}$ (which places the \mathbf{k} -module \mathfrak{g} in degree one rather than zero), in a certain sense that we will explain. Using this, Theorem 1.1.4 is a “quantization” of a standard type of statement (Proposition 3.4.4) that twisted Poisson algebra structures on the symmetric algebra $\mathrm{Sym}_{\mathbb{S}} \Sigma\mathfrak{g}$ are equivalent to twisted Lie algebra structures on $\Sigma\mathfrak{g}$. More precisely, “quantizing” this equivalence yields a strengthened theorem (4.3.1), which circumvents the need for the graded PBW theorem for \mathfrak{g} , and proves Theorem 1.1.4 (as well as the pre-Lie PBW theorem) in full generality. The proof uses Stover’s graded PBW theorem [Sto93] valid for all *connected* twisted Lie algebras, which applies to $\Sigma\mathfrak{g}$ (rather than \mathfrak{g}).

The equivalence between pre-Lie algebra structures on \mathfrak{g} and twisted Lie algebra structures on $\Sigma\mathfrak{g}$, as well as the resulting quantization procedure, generalizes from \mathbf{k} -modules \mathfrak{g} to arbitrary \mathbb{S} -modules (§4.5) and even symmetric monoidal categories (§4.7), which implies in particular quantum PBW theorems for pre-Lie algebras in these contexts (as promised in §1.4 above).

In the case of \mathbb{S} -modules, this requires passing to \mathbb{S} -bimodules. We give an alternative approach in this setting that involves taking the suspension $\Sigma\mathfrak{g}$ in a more careful way, remaining in the realm of \mathbb{S} -modules, while still implying the analogue of Theorem 1.1.4 (§4.6).

1.6. Outline of paper. — The main contributions of this paper are the following:

1. To give a simple proof of Theorem 1.1.4 (§2) using the graded PBW isomorphism (this proof does *not* require \mathbb{S} -modules or twisted algebras);
2. To point out the connection between pre-Lie algebras and twisted Lie algebras (§3), and use this to prove a strengthening of the theorem (Theorem 4.3.1) without any assumptions on \mathfrak{g} or \mathbf{k} ;
3. To generalize the above results and observations to the case where \mathfrak{g} is a twisted pre-Lie algebra, or a pre-Lie algebra in an arbitrary abelian symmetric monoidal category with limits (§§4.5–4.7);
4. To sketch a simple proof of a categorical generalization of Stover’s twisted graded PBW theorem as well as the usual graded PBW theorems in a unified context (§5).

In the appendix, we recall the PBW counterexamples from [Coh63] and remark that a pre-Lie identity [Tou06] which generalizes a classical p -th power identity of Zassenhaus explains why they do not extend to the pre-Lie setting (in accordance with Corollary 1.3.1.(i)).

⁽³⁾Twisted Lie algebras are old in topology, and predate these references.

1.7. Acknowledgements. — This work grew out of an attempt to understand and improve [OG05, OG08]. I am grateful to M. Livernet for useful discussions, as well as pointing out the main references including *op. cit.*, and for many helpful corrections and suggestions. I am grateful to M. Van den Bergh for useful discussions, and to my Ph.D. advisor, V. Ginzburg, for his guidance. I also thank M. Ronco for answering questions about [Ron07], and J.-M. Oudom for answering questions about [OG05, OG08] and providing revisions. I am very grateful to the anonymous referee for helpful suggestions, and in particular pointing out the references [Coh63, Rev77, Hig69]. Finally, I would like to thank the participants and organizers of the 2009 CIRM conference on operads for the opportunity to present this work and for their helpful comments and questions. This work was supported by the University of Chicago Mathematics Department's VIGRE grant and a five-year AIM fellowship.

2. Theorem 1.1.4 using the graded PBW theorem

This section will not require the notion of \mathbb{S} -module or twisted algebras.

Here, we prove Theorem 1.1.4 under the assumption that the graded PBW isomorphism $\mathrm{Sym} \mathfrak{g} \xrightarrow{\sim} \mathrm{gr}(U\mathfrak{g})$ holds for a given pre-Lie algebra \mathfrak{g} (e.g., if \mathfrak{g} is a free \mathbf{k} -module, $\mathbf{k} \supseteq \mathbb{Q}$, or \mathbf{k} is a Dedekind domain), by inductively constructing a lift to a coalgebra isomorphism $\mathrm{Sym} \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$, that has the needed properties.

2.1. Proof of Theorem 1.1.4. — It follows immediately from Proposition 1.2.2, specifically formula (1) in §1.2, that, given a star product $*$ as in the theorem, (1.1.6) yields a pre-Lie structure on \mathfrak{g} . Thus, it remains to show that any pre-Lie algebra (\mathfrak{g}, \circ) admits a unique star product $*$ on $\mathrm{Sym} \mathfrak{g}$ satisfying (i)–(iii). By Proposition 1.2.2 again, uniqueness is immediate, so it suffices to show that such a star product $*$ exists.

Let (\mathfrak{g}, \circ) be a pre-Lie algebra. We also let \mathfrak{g} denote the associated Lie algebra with bracket $\{x, y\} := x \circ y - y \circ x$. We prove the theorem by constructing a coalgebra isomorphism $\Phi : \mathrm{Sym} \mathfrak{g} \rightarrow U\mathfrak{g}$ (thereby simultaneously proving Corollary 1.3.1(ii)). We construct Φ inductively on degree, such that $\mathrm{gr}(\Phi)$ is the graded PBW morphism (which is assumed to be an isomorphism), and such that the induced star-product $*$ on $\mathrm{Sym} \mathfrak{g}$ satisfies (1.1.6) and (1.1.5). (One may also notice that Φ extends uniquely in each degree.)

We now begin the inductive construction of Φ . In degree 1, set $\Phi(x) = x$ for all $x \in \mathfrak{g}$. Inductively, begin with a coalgebra morphism $\Phi_{\leq n-1} : \mathrm{Sym}^{\leq n-1} \mathfrak{g} \rightarrow U\mathfrak{g}$ such that $\mathrm{gr} \Phi_{\leq n-1}$ is the graded PBW morphism, and such that the product

$$(2.1.1) \quad * : \bigoplus_{i+j \leq n-1} \mathrm{Sym}^i \mathfrak{g} \otimes \mathrm{Sym}^j \mathfrak{g} \rightarrow \mathrm{Sym}^{\leq n-1} \mathfrak{g}$$

defined by $\Phi(a * b) = \Phi(a) \cdot \Phi(b)$ satisfies (1.1.5). (Here \cdot is the product in $U\mathfrak{g}$).

We will extend $\Phi_{\leq n-1}$ to $\Phi_{\leq n} : \text{Sym}^{\leq n} \mathfrak{g} \rightarrow U\mathfrak{g}$ satisfying the same conditions. Note that, applying condition (2) of Proposition 1.2.2 repeatedly with $c \in \mathfrak{g}$,⁽⁴⁾ $\Phi_{\leq n}$ will need to satisfy

$$(2.1.2) \quad \begin{aligned} \Phi_{\leq n-1}(x_1 x_2 \cdots x_{n-1}) x_n &= \Phi_{\leq n}((x_1 x_2 \cdots x_{n-1}) * x_n) \\ &= \Phi_{\leq n}(x_1 x_2 \cdots x_n) + \Phi_{\leq n-1} \left(\sum_{i=1}^{n-1} x_1 x_2 \cdots x_{i-1} (x_i \circ x_n) x_{i+1} \cdots x_{n-1} \right), \forall x_1, x_2, \dots, x_n \in \mathfrak{g}. \end{aligned}$$

By linearity of Φ , setting the LHS to the RHS uniquely extends $\Phi_{\leq n-1}$ to $\Phi_{\leq n}$. We must check that the formula is well-defined, by showing that the resulting expression for $\Phi_{\leq n}(x_1 x_2 \cdots x_n)$ is symmetric in the variables. It is obviously symmetric in x_1, x_2, \dots, x_{n-1} , so it suffices to check that it is symmetric under permuting x_{n-1} and x_n (this is the main step of the proof).

To do this, by the induction hypothesis, (2.1.2) is equivalent to

$$(2.1.3) \quad \Phi_{\leq n-1}(x_1 x_2 \cdots x_{n-2}) x_{n-1} x_n = \Phi_{\leq n}(((x_1 x_2 \cdots x_{n-2}) * x_{n-1}) * x_n),$$

by further expanding the RHS using the formula (2) of Proposition 1.2.2 with $c \in \mathfrak{g}$. So, to prove the symmetry of x_{n-1}, x_n , it suffices to show that

$$(2.1.4) \quad \Phi_{\leq n-1}(x_1 x_2 \cdots x_{n-2}) \{x_{n-1}, x_n\} = \Phi_{\leq n}(((x_1 x_2 \cdots x_{n-2}) * x_{n-1}) * x_n - ((x_1 x_2 \cdots x_{n-2}) * x_n) * x_{n-1}),$$

for all $x_1, \dots, x_n \in \mathfrak{g}$. This may be rewritten as the claim:

$$(2.1.5) \quad (x_1 x_2 \cdots x_{n-2}) * \{x_{n-1}, x_n\} = ((x_1 x_2 \cdots x_{n-2}) * x_{n-1}) * x_n - ((x_1 x_2 \cdots x_{n-2}) * x_n) * x_{n-1}.$$

By expanding the LHS and RHS using (2.1.2) (i.e., formula (2) of Proposition 1.2.2), and substituting $\{x_{n-1}, x_n\} = x_{n-1} \circ x_n - x_n \circ x_{n-1}$, this follows from the pre-Lie identity (1.1.2).

Next, we show that $\Phi_{\leq n}$ is a morphism of coalgebras. This follows from the fact that $\Phi_{\leq n-1}$ is a morphism of coalgebras, using (for a, b homogeneous of positive degree such that $|ab| = n$)

$$(2.1.6) \quad \begin{aligned} \Delta(\Phi_{\leq n}(a * b)) &= \Delta(\Phi_{\leq n-1}(a) \Phi_{\leq n-1}(b)) = \Delta(\Phi_{\leq n-1}(a)) \Delta(\Phi_{\leq n-1}(b)) \\ &= \Phi_{\leq n-1}^{\otimes 2}(\Delta(a)) \Phi_{\leq n-1}^{\otimes 2}(\Delta(b)) = \Phi_{\leq n}^{\otimes 2}(\Delta(a) * \Delta(b)). \end{aligned}$$

It remains to show that (2.1.2) defines a product $*$ such that $\text{Sym}^i \mathfrak{g} * \text{Sym}^{n-i} \mathfrak{g} \subset \text{Sym}^{\geq i} \mathfrak{g}$. This follows by definition for $n-1 \leq i \leq n$. For every $1 \leq i \leq n-2$, it suffices to show that

$$(2.1.7) \quad (x_1 x_2 \cdots x_i) * (x_{i+1} * (x_{i+2} x_{i+3} \cdots x_n)) \in \text{Sym}^{\geq i} \mathfrak{g}, \quad \forall x_1, \dots, x_n \in \mathfrak{g}.$$

Using reverse induction on i and associativity of $*$, the statement follows immediately.

3. Pre-Lie algebras as twisted Lie algebras

In this section, we explain our main observation (Proposition 3.3.3) connecting pre-Lie algebras with twisted Lie algebras (whose definition we recall). Along the way, we will use the notion of (twisted) Poisson algebras, although in the end Proposition 3.3.3 does not require it.

⁽⁴⁾This is the only part of Proposition 1.2.2 that we need, and it also follows immediately from applying a single coproduct. That is, we will not really need the precise formula for $*$, unlike [OG05, OG08]. (Even the uniqueness of $*$ follows from the existence argument without requiring Proposition 1.2.2, if we are slightly more careful.)

3.1. Preliminaries and motivation. — Let \mathbf{k} be an arbitrary commutative ring. All unadorned tensor products, symmetric algebras, tensor algebras, and so on will be assumed to be over \mathbf{k} . We will use in this section Roman letters (e.g., V) for (graded) \mathbf{k} -modules, to avoid confusion with the \mathbb{S} -modules we will discuss in subsequent sections (which we will denote by Fraktur letters, except for twisted associative or commutative algebras).

Observe that a Lie algebra structure on a \mathbf{k} -module V is equivalent to a Poisson algebra structure of degree -1 on the commutative algebra $\mathrm{Sym} V$, i.e., a Poisson bracket $\{, \} : \mathrm{Sym} V \otimes \mathrm{Sym} V \rightarrow \mathrm{Sym} V$ satisfying

$$(3.1.1) \quad \{V, V\} \subseteq V.$$

We may consider also a homogeneous version of the above construction. Roughly, we introduce a parameter t of degree one, and define a new bracket $\{v, w\} = t \cdot \{v, w\}_{\mathrm{old}}$, for $v, w \in V$.

Precisely, let $V[t] := V \otimes \mathbf{k}[t]$ be the free $\mathbf{k}[t]$ -module generated by V . By a Lie algebra structure on $V[t]$ over $\mathbf{k}[t]$, we mean a Lie algebra structure that is $\mathbf{k}[t]$ -linear, i.e., such that $\{tf, g\} = t\{f, g\}$ for all f, g . We consider $V[t]$ as graded with $|t| = 1 = |V|$.

Next, form the commutative algebra $\mathrm{Sym}_{\mathbf{k}[t]}(V[t]) \cong \mathrm{Sym} V \otimes \mathbf{k}[t]$, equipped with the total grading such that $|t| = 1 = |V|$. We consider *graded* Poisson structures on $\mathrm{Sym}_{\mathbf{k}[t]}(V[t])$ over $\mathbf{k}[t]$ (i.e., such that t is central) which satisfy

$$(3.1.2) \quad \{V[t], V[t]\} \subseteq V[t], \quad \text{i.e., } \{V, V\} \subseteq Vt.$$

Such structures are canonically equivalent to graded Lie algebra structures on $V[t]$ over $\mathbf{k}[t]$, which are in turn the same as ordinary Lie algebra structures on V .

An alternative and useful construction is to set $\widehat{V} := V \oplus \langle t \rangle$, and notice that

$$(3.1.3) \quad \mathrm{Sym}_{\mathbf{k}[t]}(V[t]) \cong \mathrm{Sym}_{\mathbf{k}} \widehat{V}.$$

In these terms, we are interested in graded Poisson structures on $\mathrm{Sym}_{\mathbf{k}} \widehat{V}$ such that the second condition of (3.1.2) holds.

Remark 3.1.4. — Without the condition (3.1.2), graded Poisson structures on $\mathrm{Sym}_{\mathbf{k}[t]} V[t]$ over $\mathbf{k}[t]$ are the same as *filtered* Poisson brackets on $\mathrm{Sym} V$ of degree ≤ 0 . These are determined by their restriction to $V \otimes V \rightarrow \langle t^2 \rangle \oplus tV \oplus \mathrm{Sym}^2 V$, yielding a skew-symmetric form on V , a Lie bracket, and a quadratic Poisson bracket on V , satisfying certain compatibility conditions.

3.2. Twisted algebras and \mathbb{S} -modules. — A central observation of this paper is that *pre-Lie algebras arise as the twisted version of the above construction*. In this subsection we recall the needed preliminaries.

Definition 3.2.1. — An \mathbb{S} -module is a \mathbb{N} -graded \mathbf{k} -module $\mathfrak{g} = \bigoplus_{m \geq 0} \mathfrak{g}_m$ together with an action of S_m on \mathfrak{g}_m by \mathbf{k} -module automorphisms.

Note that an \mathbb{S} -module concentrated in degree zero is the same as an ordinary \mathbf{k} -module.

A “twisted” (commutative, Lie, Poisson) algebra is a (commutative, Lie, Poisson) algebra in the category of \mathbb{S} -modules rather than \mathbf{k} -modules. To make this precise, one equips the category of \mathbb{S} -modules with the structure of a symmetric monoidal category (see, e.g., [JS93]), using the following well known formulas:

$$(3.2.2) \quad \mathfrak{g} \otimes_{\mathbb{S}} \mathfrak{h} := \bigoplus_p \bigoplus_{m+n=p} \mathrm{Ind}_{S_m \times S_n}^{S_p} \mathfrak{g}_m \otimes \mathfrak{h}_n,$$

$$(3.2.3) \quad \beta : \mathfrak{g} \otimes_{\mathbb{S}} \mathfrak{h} \xrightarrow{\sim} \mathfrak{h} \otimes_{\mathbb{S}} \mathfrak{g}, \quad \beta(g \otimes h) = (12)^{|h|, |g|}(h \otimes g),$$

where $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$ are homogeneous of degrees $|g|$ and $|h|$, respectively, and $(12)^{|h|,|g|} \in S_{|g|+|h|}$ is the permutation which swaps the two blocks $\{1, 2, \dots, |h|\}$ and $\{|h|+1, |h|+2, \dots, |g|+|h|\}$, i.e.,

$$(3.2.4) \quad (12)^{|h|,|g|}(a) = \begin{cases} a + |g|, & \text{if } 1 \leq a \leq |h|, \\ a - |h|, & \text{if } |h| + 1 \leq a \leq |g| + |h|. \end{cases}$$

Then, a twisted (commutative, Lie, Poisson) algebra can be defined by first rewriting the usual definition in terms of binary operations $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying certain diagrams, and then replacing all tensor products by $\otimes_{\mathbb{S}}$. For example, a twisted commutative algebra is an \mathbb{S} -module A equipped with a binary operation $\mu : A \otimes_{\mathbb{S}} A \rightarrow A$ which is an \mathbb{S} -module morphism, is associative, and commutative in the sense that $\mu = \mu \circ \beta$. Without using symmetric monoidal categories, we may write the necessary definitions explicitly as follows:

Notation 3.2.5. — For brevity, all explicit elements we write of \mathbb{S} -modules will be assumed to be homogeneous without saying so. In particular, whenever we write $|x|$ for x an element of an \mathbb{S} -module, x is assumed to be homogeneous (of degree $|x|$).

Definition 3.2.6. — A twisted associative algebra is an \mathbb{S} -module $A = \bigoplus_{m \geq 0} A_m$ which is also a graded associative algebra such that $A_m \otimes A_n \rightarrow A_{m+n}$ is a morphism of $S_m \times S_n \subseteq S_{m+n}$ -modules.

A twisted commutative algebra is a twisted associative algebra such that the multiplication satisfies the identity

$$(3.2.7) \quad xy = (12)^{|y|,|x|}yx.$$

A twisted Lie algebra is an \mathbb{S} -module \mathfrak{g} equipped with a binary operation $\{, \} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$$(3.2.8) \quad \{x, y\} = -(12)^{|y|,|x|}\{y, x\},$$

$$(3.2.9) \quad \{x, \{y, z\}\} + (123)^{|y|,|z|,|x|}\{y, \{z, x\}\} + (132)^{|z|,|x|,|y|}\{z, \{x, y\}\} = 0.$$

A twisted Poisson algebra is an \mathbb{S} -module which is equipped with both a twisted commutative and a twisted Lie algebra structure, satisfying the Leibniz rule,

$$(3.2.10) \quad \{xy, z\} = x\{y, z\} + (12)^{|y|,|x|,|z|}y\{x, z\}.$$

In (3.2.9) and (3.2.10) we used the following generalization of (3.2.4):

Notation 3.2.11. — Given any $i_1, i_2, \dots, i_n \geq 0$ with sum $i_1 + \dots + i_n = r$, and any permutation $\sigma \in S_n$, define the element $\sigma^{i_1, i_2, \dots, i_n} \in S_r$ to be σ applied to the blocks $[1, i_1], [i_1 + 1, i_1 + i_2], \dots, [i_1 + i_2 + \dots + i_{n-1} + 1, i_1 + i_2 + \dots + i_n = r]$. Precisely, $\sigma : [1, r] \rightarrow [1, r]$ is the permutation sending each interval $[i_1 + \dots + i_{j-1} + 1, i_1 + \dots + i_j]$ onto $[i_{\sigma^{-1}(1)} + i_{\sigma^{-1}(2)} + \dots + i_{\sigma^{-1}(\sigma(j)-1)} + 1, i_{\sigma^{-1}(1)} + i_{\sigma^{-1}(2)} + \dots + i_{\sigma^{-1}(\sigma(j)-1)} + i_j]$, preserving order.

In other words, $(12)^{i,j,k} = (12)^{i,j} \times \text{Id}_k$, $(23)^{i,j,k} = \text{Id}_i \times (23)^{j,k}$, $(123)^{i,j,k} = (12)^{i,k,j} (23)^{i,j,k}$, $(132)^{i,j,k} = (23)^{j,i,k} (12)^{i,j,k}$, and so forth.

3.3. Twisted Lie algebras and pre-Lie algebras. — In this subsection we explain the connection between pre-Lie algebras and twisted Lie algebras. Heuristically, pre-Lie algebras are equivalent to “twisted Lie algebras concentrated in degree one.” As stated, this doesn’t make sense because any *graded*, let alone twisted, (Lie) algebra concentrated in degree one is trivial; we will fix this by introducing a parameter t as before.

Given an \mathbb{S} -module \mathfrak{g} concentrated in degree zero, i.e., just a \mathbf{k} -module $V := \mathfrak{g}_0$, one may form a corresponding \mathbb{S} -module concentrated in *degree one*, $\Sigma\mathfrak{g}$, given by $(\Sigma\mathfrak{g})_1 = V$ and $(\Sigma\mathfrak{g})_m = 0$ for $m \neq 1$. We call this the *suspension* of \mathfrak{g} . Given $x \in \mathfrak{g}$, we abusively denote the corresponding element of $\Sigma\mathfrak{g}$ also by x .

One reason why this suspension is useful is the interesting but simple fact that, if \mathfrak{h} is an \mathbb{S} -module concentrated in degree one, e.g., $\mathfrak{h} = \Sigma\mathfrak{g}$ as above, then

$$(3.3.1) \quad \text{Sym}_{\mathbb{S}} \mathfrak{h} \cong T_{\mathbf{k}} \mathfrak{h},$$

where the notation $T_{\mathbf{k}}$ means that we are taking the tensor algebra *in the category of \mathbf{k} -modules*, using the standard twisted-commutative structure via permutation of components.

As we pointed out, $\Sigma\mathfrak{g}$ cannot admit a nontrivial binary operation. To fix this, we add a parameter t , similarly to §3.1. Namely, rather than considering twisted Lie algebra structures on $\Sigma\mathfrak{g}$ itself, we consider structures on

$$(3.3.2) \quad \Sigma\mathfrak{g}[t] := \Sigma\mathfrak{g} \otimes_{\mathbb{S}} \mathbf{k}[t] \cong \mathbf{k}[t] \otimes \Sigma\mathfrak{g} \otimes \mathbf{k}[t],$$

which is the module over the twisted-commutative algebra $\mathbf{k}[t] = \text{Sym}_{\mathbb{S}} \langle t \rangle$ freely generated by $\Sigma\mathfrak{g}$. By definition, $\Sigma\mathfrak{g}[t]$ is an \mathbb{S} -module with $|t| = 1 = |\Sigma\mathfrak{g}|$.

A twisted Lie algebra structure on $\Sigma\mathfrak{g}[t]$ *over $\mathbf{k}[t]$* is, by definition, a twisted Lie bracket that is $\mathbf{k}[t]$ -linear, i.e., such that $\{tf, g\} = t\{f, g\}$ for all f, g . We may now state our main observation (which makes precise the heuristic equivalence between pre-Lie algebras and “twisted Lie algebras concentrated in degree one”):

Proposition 3.3.3. — *Twisted Lie algebra structures on $\Sigma\mathfrak{g}[t]$ over $\mathbf{k}[t]$ are canonically equivalent to pre-Lie structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ on \mathfrak{g} . The equivalence is given by*

$$(3.3.4) \quad \{x, y\} = (x \circ y)t - t(y \circ x), \quad \forall x, y \in \mathfrak{g}.$$

Moreover, such $\Sigma\mathfrak{g}[t]$ are in fact the associated Lie algebras of twisted pre-Lie algebras.

For the final statement, twisted pre-Lie algebras are defined as pre-Lie algebras in the category of \mathbb{S} -modules; explicitly, the twisted pre-Lie axiom is

$$(3.3.5) \quad (x \circ y) \circ z - x \circ (y \circ z) = (23)^{|x|, |z|, |y|}((x \circ z) \circ y - x \circ (z \circ y)).$$

Proof. — Begin with a twisted Lie algebra structure on $\Sigma\mathfrak{g}[t]$. By restricting to $\Sigma\mathfrak{g}$, we obtain a map

$$(3.3.6) \quad \Sigma\mathfrak{g} \otimes \Sigma\mathfrak{g} \rightarrow ((\Sigma\mathfrak{g})t \oplus t(\Sigma\mathfrak{g})), \quad \{v, w\} = (x \circ y)t - t(y \circ x).$$

The map \circ is equivalent to the bracket $\{, \}$ using the formula

$$(3.3.7) \quad \{t^a x t^b, t^c y t^d\} = t^a (x \circ y) t^{b+c+d} - t^{a+b+c} (y \circ x) t^d.$$

We conclude that twisted Lie structures on $\Sigma\mathfrak{g}[t]$ over $\mathbf{k}[t]$ are the same as binary operations $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ such that the bracket defined by (3.3.7) satisfies (3.2.9). It is easy to see that (3.2.9) holds for this bracket if and only if it holds whenever $x, y, z \in \Sigma\mathfrak{g}$. In this case, the LHS of (3.2.9) lives in $(\Sigma\mathfrak{g})t^2 \oplus t(\Sigma\mathfrak{g})t \oplus t^2(\Sigma\mathfrak{g})$. Each component of the resulting identity is easily seen to be equivalent to the pre-Lie condition (1.1.2).

For the final statement, one defines the twisted pre-Lie structure on $\Sigma\mathfrak{g}[t]$ by $(t^a v t^b) \circ (t^c w t^d) = t^a (v \circ w) t^{b+c+d}$. It follows from the above analysis that this is a twisted pre-Lie structure. \square

Remark 3.3.8. — One can also argue slightly differently to prove equivalence of the twisted Jacobi identity for $\Sigma\mathfrak{g}[t]$ and the pre-Lie identity for \mathfrak{g} : for (3.3.7) to define a twisted Lie bracket, the commutator $\{x, y\} := x \circ y - y \circ x$ must be a Lie bracket by setting $t = 1$, and then each component of (3.2.9) becomes the condition that \circ is a right Lie action of $(\mathfrak{g}, \{, \})$ on itself. As remarked by M. Livernet, this is well known to be equivalent to the pre-Lie condition.

3.4. Twisted Poisson algebras. — We first make some definitions we will need for the rest of the paper. Let

$$(3.4.1) \quad \text{Sym}_{\mathbb{S}, \mathbf{k}[t]} \Sigma\mathfrak{g}[t] := \text{Sym}_{\mathbb{S}} \Sigma\mathfrak{g}[t] / (t \otimes f - tf)$$

be the twisted-commutative algebra over $\mathbf{k}[t]$ generated by the $\mathbf{k}[t]$ -module $\Sigma\mathfrak{g}[t]$. Here, the quotient is by the *twisted-commutative ideal* generated by $t \otimes f - tf$. Analogously to (3.1.3), and using (3.3.1), we have the formula

$$(3.4.2) \quad \text{Sym}_{\mathbb{S}, \mathbf{k}[t]} \Sigma\mathfrak{g}[t] \cong \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}} \cong T_{\mathbf{k}} \widehat{\mathfrak{g}},$$

where $\text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}$ is equipped with its usual structure of twisted-commutative algebra, and

$$(3.4.3) \quad \widehat{\mathfrak{g}} := \Sigma\mathfrak{g} \oplus \langle t \rangle,$$

again with $|t| = 1$.

Now, we explain a “quasiclassical” analogue of Theorem 4.3.1 of the next section (the result which implies Theorem 1.1.4 in full generality), which can be regarded as a translation of Proposition 3.3.3 to the Poisson (rather than Lie) setting. This will not be needed for the rest of the paper, so the reader can skip it if desired.

Proposition 3.4.4. — *Twisted Poisson structures on $T_{\mathbf{k}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbb{S}, \mathbf{k}[t]}(\Sigma\mathfrak{g}[t])$ over $\mathbf{k}[t]$ satisfying*

$$(3.4.5) \quad \{\Sigma\mathfrak{g}[t], \Sigma\mathfrak{g}[t]\} \subseteq \Sigma\mathfrak{g}[t], \quad \text{i.e., } \{\Sigma\mathfrak{g}, \Sigma\mathfrak{g}\} \subseteq (t(\Sigma\mathfrak{g}) \oplus (\Sigma\mathfrak{g})t),$$

are equivalent to pre-Lie algebra structures \circ on \mathfrak{g} , by (3.3.4).

Proof. — We need to show that twisted Poisson structures on $T_{\mathbf{k}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbb{S}, \mathbf{k}[t]}(\Sigma\mathfrak{g}[t])$ over $\mathbf{k}[t]$ satisfying (3.4.5) are equivalent to twisted Lie algebra structures on $\Sigma\mathfrak{g}[t]$. This follows as in the ordinary setting: any twisted Poisson structure on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$ restricts to a twisted Lie algebra structure on $\Sigma\mathfrak{g}[t]$; conversely, there is a unique extension of any twisted Lie algebra structure on $\Sigma\mathfrak{g}[t]$ satisfying (3.4.5) to a twisted Poisson structure on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$. The latter result is an easy special case of the fact (see [Sch09, Proposition 1.10]) that, for every \mathbb{S} -module \mathfrak{h} , twisted Poisson structures on $\text{Sym}_{\mathbb{S}} \mathfrak{h}$ are equivalent to binary operations $\mathfrak{h} \otimes_{\mathbb{S}} \mathfrak{h} \rightarrow \text{Sym}_{\mathbb{S}} \mathfrak{h}$ satisfying (3.2.8) and (3.2.9) for $x, y, z \in \mathfrak{h}$, and similarly when \mathfrak{h} is over $\mathbf{k}[t]$. The condition (3.4.5) guarantees that the binary operation $\Sigma\mathfrak{g}[t] \otimes \Sigma\mathfrak{g}[t] \rightarrow \text{Sym}_{\mathbb{S}, \mathbf{k}[t]}(\Sigma\mathfrak{g}[t])$ in fact lands in $\Sigma\mathfrak{g}[t]$, which is therefore a twisted Lie bracket. \square

4. Generalizations and full proof of Theorem 1.1.4

The first goal of this section is to state and prove a generalization of Theorem 1.1.4 which yields a twisted coalgebra isomorphism $T_{\mathbf{k}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbb{S}, \mathbf{k}[t]}(\Sigma\mathfrak{g}[t]) \xrightarrow{\sim} U_{\mathbb{S}, \mathbf{k}[t]}(\Sigma\mathfrak{g}[t]) = U_{\mathbb{S}} \widehat{\mathfrak{g}}$. The resulting Theorem 4.3.1 can be viewed as a *noncommutative* analogue of Theorem 1.1.4, since it replaces $\text{Sym}_{\mathbf{k}} \mathfrak{g}$ (better, $\text{Sym}_{\mathbf{k}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbf{k}} \mathfrak{g} \otimes_{\mathbf{k}} \mathbf{k}[t]$) with $T_{\mathbf{k}} \widehat{\mathfrak{g}}$. Moreover, this theorem is, in a sense, easier to prove than the original one, since the graded PBW isomorphism will be automatic rather than an assumption, and some of the argument simplifies.

Then, we explain how to further generalize this to prove twisted analogues of Theorem 1.1.4 and Corollary 1.3.1, applicable to any *twisted pre-Lie algebra* (Theorem 4.5.1), or more generally, to any pre-Lie algebra in a suitable symmetric monoidal category (Theorem 4.7.3).

In the case of twisted pre-Lie algebras, the categorical approach requires working with \mathbb{S} -bimodules. This is not entirely satisfactory, and we give a construction entirely in \mathbb{S} -modules, by extending the suspension functor to act on arbitrary \mathbb{S} -modules, in particular taking twisted pre-Lie algebras to twisted pre-Lie algebras. This culminates in Theorem 4.6.10, which implies all the results about twisted pre-Lie algebras without using \mathbb{S} -bimodules. There is also a common generalization of this result and the categorical Theorem 4.7.3: see Remark 4.7.9.

Before proving these results, we need to recall Stover's graded PBW theorem for connected twisted Lie algebras. We also recall the notions of twisted coalgebras and bialgebras, relevant to the twisted symmetric and enveloping algebras $\text{Sym}_{\mathbb{S}} \mathfrak{g}$ and $U_{\mathbb{S}} \mathfrak{g}$. These occupy §§4.1, 4.2.

4.1. Twisted PBW theorem. — By recasting pre-Lie algebras as twisted Lie algebras, we may apply the twisted PBW theorem. As is well known (see, e.g., [Bar78]), if \mathfrak{h} is a twisted Lie algebra which is free as a \mathbf{k} -module, it satisfies the graded PBW theorem. That is, one may consider the twisted enveloping algebra

$$(4.1.1) \quad U_{\mathbb{S}} \mathfrak{h} := T_{\mathbb{S}} \mathfrak{h} / (xy - (12)^{|y|, |x|} yx - \{x, y\})_{x, y \in \mathfrak{h}},$$

where $T_{\mathbb{S}} \mathfrak{h}$ is the free twisted associative algebra generated by \mathfrak{h} , and the quotient is by a *twisted algebra* ideal (i.e., a usual ideal which is also an \mathbb{S} -submodule). Then, $U_{\mathbb{S}} \mathfrak{h}$ is again filtered using $\mathbf{k} = F_0(T_{\mathbb{S}} \mathfrak{h}) \subseteq F_1(T_{\mathbb{S}} \mathfrak{h}) = \mathbf{k} \oplus \mathfrak{h}$. Under the same assumptions as before (e.g., \mathfrak{h} is free as a \mathbf{k} -module or $\mathbf{k} \supseteq \mathbb{Q}$), the canonical epimorphism is an isomorphism

$$(4.1.2) \quad \text{Sym}_{\mathbb{S}} \mathfrak{h} \xrightarrow{\sim} \text{gr } U_{\mathbb{S}} \mathfrak{h},$$

preserving all structures. Remarkably, in the case that \mathfrak{h} is *connected*, i.e., \mathfrak{h} is concentrated in positive degrees, Stover noticed [Sto93] that the above isomorphism holds *without any hypotheses* on \mathfrak{h} and \mathbf{k} :

Theorem 4.1.3. — [Sto93] *Let \mathbf{k} be any commutative ring, and let \mathfrak{h} be any connected twisted Lie algebra over \mathbf{k} . Then, the canonical map (4.1.2) is an isomorphism.*

One rough explanation why this holds is that, for all n , the tensor algebra $T_{\mathbb{S}}^n \mathfrak{h}$ is a free S_n -module in each degree (by permutation of the \mathfrak{h} -factors), where by “free,” we mean a module of the form $\text{Ind}_{\{1\}}^{S_n} M = \mathbf{k}[S_n] \otimes_{\mathbf{k}} M$ for some M (which is *not* necessarily free as a \mathbf{k} -module). See §5 for a sketch of a simple proof.

Caution 4.1.4. — It is tempting to conclude that Theorem 4.1.3 for connected twisted Lie algebras, together with the connection of Proposition 3.3.3 between pre-Lie algebras \mathfrak{g} and connected twisted Lie algebras $\Sigma \mathfrak{g}[t]$, yields the pre-Lie graded PBW theorem (Theorem 1.3.1.(i)). However, we do not know how to conclude this. In more detail, if \mathfrak{g} is a pre-Lie algebra, one deduces from these results that $\text{Sym}_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t] \xrightarrow{\sim} \text{gr } U_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t]$. By taking S_n -coinvariants in each degree n and setting $t = 1$ (as we will do later to deduce Theorem 1.1.4 from Theorem 4.3.1), one obtains $\text{Sym } \mathfrak{g} \xrightarrow{\sim} (\text{gr } U_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t])_{\mathbb{S}}|_{t=1}$, where here for an \mathbb{S} -module $M = \bigoplus_{n \geq 0} M_n$, we denote the total coinvariants by $M_{\mathbb{S}} := \bigoplus_{n \geq 0} (M_n)_{S_n}$. However, one would instead like to have an isomorphism with $\text{gr } U \mathfrak{g} = \text{gr } ((U_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t])_{\mathbb{S}})|_{t=1}$. It is, however, not always true that taking coinvariants commutes with taking associated graded. One could fix this if one had a quantum PBW isomorphism, $\text{Sym } \mathfrak{h} \xrightarrow{\sim} U \mathfrak{h}$, but Stover only showed that such an isomorphism exists as a map of \mathbf{k} -modules, not as a map of \mathbb{S} -modules: in fact, this isomorphism

does *not* exist for all connected twisted Lie algebras, by the remark below (which also shows that $(\mathrm{gr} U_{\mathbb{S}}\mathfrak{h})_{\mathbb{S}} \not\cong \mathrm{gr}((U_{\mathbb{S}}\mathfrak{h})_{\mathbb{S}})$ for general twisted Lie algebras \mathfrak{h}). This isomorphism does exist, however, in our case $\mathfrak{h} = \Sigma \mathfrak{g}[t]$ for \mathfrak{g} a pre-Lie algebra, but we need Theorem 4.3.1 to show it.

Remark 4.1.5. — As we just mentioned, in the connected twisted case, the graded PBW isomorphism does *not* always lift to an isomorphism of \mathbb{S} -modules $\mathrm{Sym}_{\mathbb{S}} \mathfrak{h} \xrightarrow{\sim} U_{\mathbb{S}}\mathfrak{h}$ (Stover lifted it only to an isomorphism of \mathbf{k} -modules), i.e., there is no quantum PBW theorem in this setting. To see this, note that, when the graded PBW isomorphism does lift to an \mathbb{S} -module isomorphism, we can take S_n -coinvariants in each degree to obtain $\mathrm{Sym}_{\mathbf{k}}(\bigoplus_{m \geq 0} \mathfrak{h}_{S_m}) \xrightarrow{\sim} U_{\mathbf{k}}(\bigoplus_{m \geq 0} \mathfrak{h}_{S_m})$. For example, given any *ordinary* graded connected Lie algebra W , we can view it as a connected twisted algebra \widetilde{W} with *trivial* S_m actions in each degree m . Then, if the twisted graded PBW isomorphism for \widetilde{W} lifts to an \mathbb{S} -module isomorphism, resp., twisted coalgebra isomorphism, then the usual graded PBW morphism for W also lifts to a \mathbf{k} -module, resp., coalgebra isomorphism, $\mathrm{Sym}_{\mathbf{k}} W \xrightarrow{\sim} U_{\mathbf{k}} W$. In particular, the usual graded PBW theorem holds for W , $\mathrm{Sym}_{\mathbf{k}} W \xrightarrow{\sim} \mathrm{gr} U_{\mathbf{k}} W$. However, there are many examples of connected ordinary Lie algebras W for which the graded PBW theorem doesn't hold, such as the PBW counterexample [Coh63, §5] recalled in the appendix.

4.2. Twisted coalgebras and bialgebras. — A twisted commutative coalgebra A is a commutative coalgebra in the category of \mathbb{S} -modules, which explicitly means the following: A is an \mathbb{S} -module equipped with a comultiplication $A \rightarrow A \otimes_{\mathbb{S}} A$, which is coassociative:

$$(4.2.1) \quad (\mathrm{Id} \otimes_{\mathbb{S}} \Delta) \circ \Delta = (\Delta \otimes_{\mathbb{S}} \mathrm{Id}) \circ \Delta,$$

and cocommutative,

$$(4.2.2) \quad \Delta = \beta \circ \Delta,$$

where β is given in (3.2.3). We also require a counit, $\varepsilon : A \rightarrow \mathbf{k}$, where \mathbf{k} is viewed as an \mathbb{S} -module concentrated in degree zero, such that

$$(4.2.3) \quad (\varepsilon \otimes_{\mathbb{S}} \mathrm{Id}) \circ \Delta = \mathrm{Id} = (\mathrm{Id} \otimes_{\mathbb{S}} \varepsilon) \circ \Delta.$$

The twisted symmetric and enveloping algebras $\mathrm{Sym}_{\mathbb{S}} \mathfrak{h}$ and $U_{\mathbb{S}}\mathfrak{h}$ are equipped with a standard twisted cocommutative coproduct, which is the unique \mathbb{S} -module extension of

$$(4.2.4) \quad \Delta(x_1 x_2 \cdots x_m) = \sum_{I \sqcup J = \{1, 2, \dots, m\}} (\sigma_{I, J})^{|x_1|, \dots, |x_m|} \prod_{i \in I} x_i \otimes \prod_{j \in J} x_j, \quad \forall x_1, \dots, x_m \in \mathfrak{h},$$

where $\sigma_{I, J} \in S_m$ is the permutation which reorders the indices of the x_i 's that appear into increasing order (i.e., taking the products over I and J to be in increasing order, this is the appropriate $(|I|, |J|)$ -shuffle). The counit ε is the quotient by the augmentation ideal (\mathfrak{g}) .

A clearer, although less explicit, way to define the above coproduct is to note that $\mathrm{Sym}_{\mathbb{S}} \mathfrak{h}$ and $U_{\mathbb{S}}\mathfrak{h}$ are in fact *twisted bialgebras*, which means that Δ and ε are morphisms of twisted associative algebras, or equivalently that the multiplication and unit are morphisms of twisted coassociative coalgebras. Explicitly,

$$(4.2.5) \quad \Delta(xy) = \Delta(x) \cdot \Delta(y), \quad \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \quad \varepsilon(1) = 1,$$

where the multiplication \cdot on $A \otimes_{\mathbb{S}} A$ is given by

$$(4.2.6) \quad (u \otimes v) \cdot (x \otimes y) = (23)^{|u|, |x|, |v|, |y|} (ux \otimes vy), \quad \forall u, v, x, y \in A.$$

Then, (4.2.4) is the unique extension of $\Delta(x) = 1 \otimes x + x \otimes 1, x \in \mathfrak{h}$, from \mathfrak{h} to all of $\mathrm{Sym}_{\mathbb{S}} \mathfrak{h}$ or $U_{\mathbb{S}}\mathfrak{h}$.

In the $\mathbf{k}[t]$ -module case, one similarly defines twisted-commutative coproducts on $\text{Sym}_{\mathbb{S}, \mathbf{k}[t]} \mathfrak{h}$ and $U_{\mathbb{S}, \mathbf{k}[t]} \mathfrak{h}$, yielding twisted bialgebras over $\mathbf{k}[t]$. Being “over $\mathbf{k}[t]$ ” here refers to the fact that $\mathbf{k}[t]$ is a sub-bialgebra ($\mathbf{k}[t]$ itself is a twisted bialgebra via $\mathbf{k}[t] = \text{Sym}_{\mathbb{S}} \langle t \rangle$, i.e., the usual graded bialgebra $\mathbf{k}[t]$ with $|t| = 1$ equipped with the trivial S_m -action in each degree m). Thus, specializing to (3.3.2), $T_{\mathbf{k}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t]$ and $U_{\mathbb{S}} \widehat{\mathfrak{g}} := U_{\mathbb{S}, \mathbf{k}[t]} \Sigma \mathfrak{g}[t]$ are twisted cocommutative bialgebras.

4.3. “Noncommutative” generalization of Theorem 1.1.4. — We will use the filtration on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$ generated by the $\{0, 1\}$ -degree filtration on $\widehat{\mathfrak{g}}$: $\langle t \rangle = F_0(\widehat{\mathfrak{g}}) \subseteq F_1(\widehat{\mathfrak{g}}) = \widehat{\mathfrak{g}}$.

Theorem 4.3.1. — *Let \mathbf{k} be any commutative ring and \mathfrak{g} any \mathbf{k} -module. Star products $*$ on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$ satisfying the conditions*

- (i) $*$ forms a twisted bialgebra with the usual coproduct Δ ,
- (ii) $*$ is a filtered product whose associated graded is the usual product on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$,
- (iii) $*$ satisfies

$$(4.3.2) \quad (T_{\mathbf{k}}^m(\Sigma \mathfrak{g})) * T_{\mathbf{k}} \widehat{\mathfrak{g}} \subseteq (T_{\mathbf{k}}^m(\Sigma \mathfrak{g}))(T_{\mathbf{k}} \widehat{\mathfrak{g}}),$$

- (iv) $f * t = ft$ and $t * f = tf$, for all $f \in T_{\mathbf{k}} \widehat{\mathfrak{g}}$,

are equivalent to (right) pre-Lie algebra structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, under the correspondence

$$(4.3.3) \quad x * y = xy + (x \circ y)t, \quad \forall x, y \in \mathfrak{g}.$$

We remark that, using (ii) and (iii), condition (iv) is equivalent to the condition that t is twisted-central in $(T_{\mathbf{k}} \widehat{\mathfrak{g}}, *)$. Furthermore, we need not mention (iv) or the words “pre-Lie” in the theorem, if we say instead that star products satisfying (i)–(iii) are equivalent to twisted Lie algebra structures on $\Sigma \mathfrak{g}[t]$ over $\mathbf{k}[t]$ (or twisted Poisson structures on $T_{\mathbf{k}} \widehat{\mathfrak{g}}$ over $\mathbf{k}[t]$).

Sketch of proof. — The proof is an adaptation of the proof of Theorem 1.1.4. By a straightforward analogue of Proposition 1.2.2, it again suffices to show that \circ gives rise to a star product $*$ satisfying (i)–(iv).

1. First, extend \circ to an operation on $\widehat{\mathfrak{g}}$ such that $t \circ x = 0 = x \circ t$ for all $x \in \widehat{\mathfrak{g}}$. With this, (4.3.3) is valid for $x, y \in \widehat{\mathfrak{g}}$, using (iv).
2. We inductively construct a twisted coalgebra isomorphism $\Phi : T_{\mathbf{k}} \widehat{\mathfrak{g}} \xrightarrow{\sim} U_{\mathbb{S}} \widehat{\mathfrak{g}}$, replacing (2.1.2) with

$$(4.3.4) \quad \begin{aligned} \Phi_{\leq n}(x_1 x_2 \cdots x_{n-1} x_n) \\ = \Phi_{\leq n-1}(x_1 x_2 \cdots x_{n-1}) x_n - \Phi_{\leq n-1} \left(\sum_{i=1}^n x_1 x_2 \cdots x_{i-1} (x_i \circ x_n) x_{i+1} \cdots x_{n-1} \right) t. \end{aligned}$$

3. Since $\text{gr } \Phi_{\leq n}$ is the twisted graded PBW isomorphism (the $\mathbf{k}[t]$ -analogue of Theorem 4.1.3 for $\mathfrak{h} = \Sigma \mathfrak{g}[t]$, obtained from the graded PBW isomorphism for the latter by imposing $\mathbf{k}[t]$ -linearity), (4.3.4) certainly gives a well-defined \mathbf{k} -linear isomorphism $T_{\mathbf{k}} \widehat{\mathfrak{g}} \xrightarrow{\sim} U_{\mathbb{S}} \widehat{\mathfrak{g}}$, and it remains to check that this is actually a twisted coalgebra morphism. We may again extend the star product to an operation $T^{\leq i} \widehat{\mathfrak{g}} \otimes T^{\leq n-i} \widehat{\mathfrak{g}} \rightarrow T^{\leq n} \widehat{\mathfrak{g}}$ such that $\Phi_{\leq n}(a * b) = \Phi_{\leq n}(a) \cdot \Phi_{\leq n}(b)$.
4. To check that $\Phi_{\leq n}$ is an \mathbb{S} -module morphism, it is enough by induction to show that $\Phi_{\leq n}$ commutes with $(n-1, n)$, or equivalently, that (similarly to (2.1.5))

$$(4.3.5) \quad (n-1, n)(x_1 \cdots x_{n-2} * x_n) * x_{n-1} = ((x_1 \cdots x_{n-2}) * x_{n-1}) * x_n - (x_1 \cdots x_{n-2}) * \{x_{n-1}, x_n\}.$$

As before, this follows by expanding $*$ using (4.3.4) and the pre-Lie identities for \circ .

5. To check that $\Phi_{\leq n}$ is a twisted coalgebra morphism, we use (2.1.6).

6. Finally, (4.3.2) is proved by replacing (2.1.7) with

$$(4.3.6) \quad (x_1 x_2 \cdots x_m) * (x_{m+1} * (x_{m+2} \cdots x_n)) \in (T_{\mathbf{k}}^m(\Sigma \mathfrak{g}))(T_{\mathbf{k}}^{n-m} \widehat{\mathfrak{g}}). \quad \square$$

4.4. General proof of Theorem 1.1.4. — Here, we deduce Theorem 1.1.4 from Theorem 4.3.1, thereby proving the former without any hypotheses on \mathfrak{g} or \mathbf{k} .

Proof of Theorem 1.1.4. — We take S_m -coinvariants in each \mathbb{S} -module degree m , and set t equal to 1. This transforms $\text{Sym}_{\mathbb{S}} \widehat{g}$ into $\text{Sym}_{\mathbf{k}} \mathfrak{g}$ and $U_{\mathbb{S}} \widehat{g}$ into $U_{\mathbf{k}} \mathfrak{g}$, where \mathfrak{g} is viewed as an ordinary Lie algebra with bracket $\{x, y\} = x \circ y - y \circ x$. Thus, any pre-Lie multiplication on \mathfrak{g} gives rise to a star product satisfying the needed conditions. Uniqueness and the converse follow from Proposition 1.2.2, as explained in §2.⁽⁵⁾ \square

4.5. Twisted generalization of the main results. — If we had worked originally in the category of \mathbb{S} -modules rather than \mathbf{k} -modules, everything we did generalizes. In particular, Theorem 1.1.4 generalizes to:

Theorem 4.5.1. — *Let \mathfrak{g} be any \mathbb{S} -module. Star products $*$ on $\text{Sym}_{\mathbb{S}} \mathfrak{g}$ satisfying the conditions*

- (i) *$*$ forms a bialgebra with the usual twisted coproduct Δ ,*
- (ii) *$*$ is a filtered product whose associated graded is the usual product on $\text{Sym}_{\mathbb{S}} \mathfrak{g}$,*
- (iii) *$*$ satisfies*

$$(4.5.2) \quad (\text{Sym}_{\mathbb{S}}^m \mathfrak{g}) * \text{Sym}_{\mathbb{S}} \mathfrak{g} \subseteq (\text{Sym}_{\mathbb{S}}^m \mathfrak{g})(\text{Sym}_{\mathbb{S}} \mathfrak{g}) = \text{Sym}_{\mathbb{S}}^{\geq m} \mathfrak{g},$$

are equivalent to (right) twisted pre-Lie algebra structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, under the correspondence

$$(4.5.3) \quad x * y = xy + x \circ y, \quad \forall x, y \in \mathfrak{g}.$$

In this context, Proposition 1.2.2 goes through in exactly the same way, yielding the formulas

- (0) $a \circ 1 = a$,
- (1) $a \circ (bx) = (a \circ b) \circ x - a \circ (b \circ x), \quad \forall a, b \in \text{Sym}_{\mathbb{S}} \mathfrak{g}, x \in \mathfrak{g}$,
- (2) $(ab) \circ c = (23)^{|a|, |c'|, |b|, |c''|} (a \circ c') (b \circ c'')$,
- (3) $a * b = (a \circ b') b''$.

As before, we deduce the twisted graded pre-Lie PBW theorem:

Corollary 4.5.4. — *If \mathfrak{g} is a twisted pre-Lie algebra, and $U_{\mathbb{S}} \mathfrak{g}$ the universal enveloping algebra of the associated twisted Lie algebra, then*

- 1. *The canonical morphism $\text{Sym}_{\mathbb{S}} \mathfrak{g} \rightarrow \text{gr}(U_{\mathbb{S}} \mathfrak{g})$ is an isomorphism;*
- 2. *The canonical morphism lifts to a twisted coalgebra isomorphism $\text{Sym}_{\mathbb{S}} \mathfrak{g} \xrightarrow{\sim} U_{\mathbb{S}} \mathfrak{g}$.*

In fact, the theorem generalizes to the case where \mathfrak{g} is a pre-Lie algebra in an *arbitrary* abelian symmetric monoidal category with limits, as we will explain in §4.7.

To prove the theorem, we once again need to be given that $\text{Sym}_{\mathbb{S}} \mathfrak{g} \rightarrow \text{gr}(U_{\mathbb{S}} \mathfrak{g})$ is an isomorphism: this is Stover's result if \mathfrak{g} is connected (or more generally if \mathfrak{g}_0 is a free \mathbf{k} -module, cf. Theorem 5.0.10.(iii)). In the general setting, we can again circumvent the need for this

⁽⁵⁾Alternatively, since we now have the PBW theorem for pre-Lie algebras, one can use a careful version of the proof in §2, which can be modified to avoid the use of Proposition 1.2.2 as pointed out in the footnote there.

assumption by proving a $\mathbf{k}[t]$ -analogue of the above (generalizing Theorem 4.3.1) and applying it to $\Sigma\mathfrak{g}[t]$.

However, à priori, $\Sigma\mathfrak{g}[t]$ is in the category of \mathbb{S} -bimodules, i.e., *bigraded* vector spaces with actions of $S_m \times S_n$ in bidegree (m, n) : the second grading comes from the suspension. The twisted graded PBW theorem (Theorem 4.1.3) generalizes to connected \mathbb{S} -bimodules. However, these arguments seem to be just as clear when one replaces \mathbb{S} -modules by an arbitrary symmetric monoidal category \mathcal{C} , and \mathbb{S} -bimodules by the category $\mathbb{S}_{\mathcal{C}}$ of symmetric sequences of objects of \mathcal{C} , as we will do in §4.7 below.

There is an alternative approach that remains in the category of \mathbb{S} -modules: to define the suspension Σ as a functor from \mathbb{S} -modules to itself, raising degrees by one. We then obtain an equivalence between twisted pre-Lie algebras \mathfrak{g} and certain twisted Lie algebra structures on $\Sigma\mathfrak{g}[t]$. We carry this through in the next section, which implies a “noncommutative” (better, “suspended”) generalization, Theorem 4.6.10, of the above.

4.6. Suspensions of arbitrary twisted pre-Lie algebras. — Here, we explain how to extend the suspension Σ as an operation on \mathbb{S} -modules, which allows us to generalize Proposition 3.3.3 and Theorem 4.3.1, and also gives one way to prove the results of the previous section, remaining in the category of \mathbb{S} -modules rather than \mathbb{S} -bimodules.

The main idea is to interpret $\Sigma\mathfrak{g}$, when \mathfrak{g} is a \mathbf{k} -module, as $\text{Ind}_{S_0 \times S_1}^{S_1} V$, where the \mathbf{k} -module \mathfrak{g} is viewed also as an \mathbb{S} -module concentrated in degree zero. Then, given an arbitrary \mathbb{S} -module \mathfrak{g} , define the \mathbb{S} -module $\Sigma\mathfrak{g}$ by

$$(4.6.1) \quad (\Sigma\mathfrak{g})_0 = 0, \quad (\Sigma\mathfrak{g})_{m+1} := \text{Ind}_{S_m \times S_1}^{S_{m+1}} \mathfrak{g}_m.$$

We will use the decomposition as \mathbf{k} -modules,

$$(4.6.2) \quad (\Sigma\mathfrak{g})_{m+1} = \mathfrak{g}_m \oplus \bigoplus_{i=1}^m (i, i+1, \dots, m+1) \mathfrak{g}_m.$$

Given an element $x \in \mathfrak{g}_m$, let

$$(4.6.3) \quad x^{(0)} := x \in (\Sigma\mathfrak{g})_{m+1}, \quad x^{(i)} := (i, i+1, \dots, m+1)x \in (\Sigma\mathfrak{g})_{m+1}.$$

Heuristically, think of the element $x^{(0)}$ as “ xs ” where we multiply x by a parameter s , with $|s| = 1$. Similarly, $x^{(1)}$ can be thought of as “ sx ”, and $x^{(i)}$ can be thought of as obtained from x by “inserting an s between the $(i-1)$ -th and i -th S_m -module components,” for $2 \leq i \leq m$.

Now, we associate to any binary operation \circ on \mathfrak{g} a twisted skew-symmetric bracket on $\Sigma\mathfrak{g}[t] := \Sigma\mathfrak{g} \otimes_{\mathbb{S}} \mathbf{k}[t]$, such that, heuristically, $\{x^{(i)}, y^{(j)}\}$ contains an s and a t in the first and second places where an s appears in $x^{(i)} \otimes y^{(j)}$, i.e., in the i -th and $(|x|+1+j)$ -th components in the case $i, j \geq 1$. We state this precisely for $i = j = 0$ by

$$(4.6.4) \quad \{x^{(0)}, y^{(0)}\} = (x \circ y)^{(|x|)} t - (12)^{|y|+1, |x|+1} (y \circ x)^{(|y|)} t,$$

which heuristically may be written as

$$(4.6.5) \quad \{xs, ys\} = (x \circ^s y) t - (12)^{|y|+1, |x|+1} (y \circ^s x) t.$$

Another motivation for these formulas is that they describe the quasiclassical limit of star products of the form (4.6.12) we consider below (generalizing (4.3.3)).

In the original situation where \mathfrak{g} is concentrated in degree zero, (4.6.4) reduces to (3.3.4). More generally, Proposition 3.3.3 extends to

Proposition 4.6.6. — *The bracket $\{, \}$ of (4.6.4) defines a twisted Lie algebra structure on $\Sigma \mathfrak{g}[t]$ over $\mathbf{k}[t]$ if and only if \circ defines a twisted pre-Lie algebra structure on \mathfrak{g} . In this case, $\Sigma \mathfrak{g}[t]$ is itself a twisted pre-Lie algebra, under the $\mathbf{k}[t]$ -linear operation*

$$(4.6.7) \quad x^{(0)} \circ y^{(0)} := (x \circ y)^{(|x|)} t, \quad t \circ u = 0 = u \circ t.$$

In the proposition, $\mathbf{k}[t]$ -linearity means that $(ut) \circ v = u \circ (tv)$, $t(u \circ v) = (tu) \circ v$, and $(u \circ v)t = u \circ (vt)$, for all u, v .

Proof. — We need to show that, given a binary operation \circ on \mathfrak{g} and a bracket $\{, \}$ on $\Sigma \mathfrak{g} \otimes_{\mathbb{S}} \mathbf{k}[t]$ satisfying (4.6.4), the twisted pre-Lie identity (3.3.5) for \mathfrak{g} is equivalent to the twisted Jacobi identity (3.2.9) for $\Sigma \mathfrak{g} \otimes_{\mathbb{S}} \mathbf{k}[t]$. It suffices to consider the twisted Jacobi identity for elements of the form $x^{(0)}, y^{(0)}, z^{(0)}$, where x, y, z are homogeneous elements of \mathfrak{g} . Then,

$$(4.6.8) \quad \begin{aligned} & \{x^{(0)}, \{y^{(0)}, z^{(0)}\}\} + (132)^{|z|+1, |x|+1, |y|+1} \{z^{(0)}, \{x^{(0)}, y^{(0)}\}\} + (123)^{|y|+1, |z|+1, |x|+1} \{y^{(0)}, \{z^{(0)}, x^{(0)}\}\} \\ &= (34)^{|x|+1, |y|, |z|, 1, 1} (x \circ (y \circ z) - (x \circ y) \circ z - (23)^{|x|, |z|, |y|} (x \circ (z \circ y) - (x \circ z) \circ y))^{(|x|)} tt + \dots, \end{aligned}$$

where \dots are the terms obtained from the given ones by replacing (x, y, z) with (z, x, y) and applying $(132)^{|z|+1, |x|+1, |y|+1}$, and by performing this operation twice. The equivalence of the proposition follows immediately from this formula.

To show that (4.6.7) is a pre-Lie structure is an explicit verification: it suffices to consider the pre-Lie identity for a triple of elements $x^{(0)}, y^{(0)}, z^{(0)} \in \Sigma \mathfrak{g}$ obtained from $x, y, z \in \mathfrak{g}$, and then the identity follows from the pre-Lie identity for the triple x, y, z with a little bit of extra bookkeeping. In more detail, $x^{(0)} \circ (y^{(0)} \circ z^{(0)}) = (34)^{|x|+1, |y|, |z|, 1, 1} (x \circ (y \circ z))^{(|x|)} tt$, and similarly the other three terms of the pre-Lie identity for $(x^{(0)}, y^{(0)}, z^{(0)})$ can be obtained from the corresponding terms of the identity for (x, y, z) by the map $T \mapsto (34)^{|x|+1, |y|, |z|, 1, 1} T^{(x)} tt$. \square

Finally, as before, Theorem 4.5.1 has the following stronger “noncommutative” analogue. Let us again use the notation (3.4.3) and consider the algebras

$$(4.6.9) \quad \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}} = \text{Sym}_{\mathbb{S}, \mathbf{k}[t]}(\Sigma \mathfrak{g}[t]), \quad U_{\mathbb{S}} \widehat{\mathfrak{g}} = U_{\mathbb{S}, \mathbf{k}[t]}(\Sigma \mathfrak{g}[t]).$$

Also, let $(\text{Sym}_{\mathbb{S}} \Sigma \mathfrak{g}) \subset \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}$ be the twisted commutative subalgebra generated by $\Sigma \mathfrak{g} \subset \widehat{\mathfrak{g}}$.

Theorem 4.6.10. — *Let \mathfrak{g} be any \mathbb{S} -module. Star products $*$ on $\text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}$ satisfying the conditions*

- (i) *$*$ forms a twisted bialgebra with the usual twisted coproduct Δ ,*
- (ii) *$*$ is a filtered product whose associated graded is the usual product on $\text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}$,*
- (iii) *$*$ satisfies*

$$(4.6.11) \quad \text{Sym}_{\mathbb{S}}^m(\Sigma \mathfrak{g}) * \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}} \subseteq \text{Sym}_{\mathbb{S}}^m(\Sigma \mathfrak{g})(\text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}),$$

- (iv) *$f * t = ft$ and $t * f = tf$ for all $f \in \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}}$,*

are equivalent to (right) twisted pre-Lie algebra structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, under the correspondence

$$(4.6.12) \quad x^{(0)} * y^{(0)} = x^{(0)} y^{(0)} + (x \circ y)^{(|x|)} t, \quad \forall x, y \in \mathfrak{g}.$$

This theorem can be proved using only Theorem 4.1.3 and Proposition 4.6.6 (see the sketch below), and implies all of the other results of Sections 1 through 4. In particular, one may use it to deduce the results of the previous subsection without using \mathbb{S} -bimodules. Also, the remarks following Theorem 4.3.1 (allowing us to replace the condition (iv) and the words “pre-Lie”), apply here as well.

Sketch of proof. — The proof is again an adaptation of the proof of Theorem 1.1.4. By a straightforward analogue of Proposition 1.2.2 using (4.6.4), it once again suffices to show that \circ gives rise to a star product $*$ satisfying (i)–(iv).

1. Thanks to (iv) and the pre-Lie structure (4.6.7) on $\Sigma\mathfrak{g}[t]$, (4.6.12) becomes

$$(4.6.13) \quad u * v = uv + u \circ v,$$

valid for all $u, v \in \widehat{\mathfrak{g}}$.

2. We inductively construct a *twisted* coalgebra isomorphism $\Phi : \text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}} \xrightarrow{\sim} U_{\mathbb{S}} \widehat{\mathfrak{g}}$, replacing (2.1.2) with

$$(4.6.14) \quad \begin{aligned} \Phi_{\leq n}(x_1 x_2 \cdots x_{n-1} x_n) \\ = \Phi_{\leq n-1}(x_1 x_2 \cdots x_{n-1}) x_n - \Phi_{\leq n-1} \left(\sum_{i=1}^n \sigma_i \cdot x_1 x_2 \cdots x_{i-1} (x_i \circ x_n) x_{i+1} \cdots x_{n-1} \right), \end{aligned}$$

using the permutation

$$(4.6.15) \quad \sigma_i := (i+1, n, n-1, n-2, \dots, i)^{|x_1|, |x_2|, \dots, |x_i|, |x_n|, |x_{i+1}|, |x_{i+2}|, \dots, |x_{n-1}|}.$$

3. Since $\text{gr } \Phi_{\leq n}$ is the twisted graded PBW isomorphism (the $\mathbf{k}[t]$ -linear quotient of the graded PBW isomorphism of Theorem 4.1.3 for $\mathfrak{h} = \Sigma\mathfrak{g}[t]$), (4.6.14) certainly gives a well-defined \mathbf{k} -linear isomorphism $\text{Sym}_{\mathbb{S}} \widehat{\mathfrak{g}} \xrightarrow{\sim} U_{\mathbb{S}} \widehat{\mathfrak{g}}$, and it remains to check that this is actually a twisted coalgebra morphism. We may again extend the star product to an operation $\text{Sym}_{\mathbb{S}}^{\leq i} \widehat{\mathfrak{g}} \otimes \text{Sym}_{\mathbb{S}}^{\leq n-i} \widehat{\mathfrak{g}} \rightarrow \text{Sym}_{\mathbb{S}}^{\leq n} \widehat{\mathfrak{g}}$ such that $\Phi_{\leq n}(a * b) = \Phi_{\leq n}(a) \cdot \Phi_{\leq n}(b)$.

4. To check that $\Phi_{\leq n}$ is an \mathbb{S} -module morphism, it is enough by induction to show that (similarly to (2.1.5))

$$(4.6.16) \quad \begin{aligned} (n-1, n)^{|x_1|, \dots, |x_{n-2}|, |x_n|, |x_{n-1}|} (x_1 \cdots x_{n-2} * x_n) * x_{n-1} \\ = ((x_1 \cdots x_{n-2}) * x_{n-1}) * x_n - (x_1 \cdots x_{n-2}) * \{x_{n-1}, x_n\}. \end{aligned}$$

As before, this follows by expanding $*$ using (4.6.14) and the pre-Lie identities for \circ .

5. To check that $\Phi_{\leq n}$ is a twisted coalgebra morphism, we use (2.1.6).

6. Finally, (4.5.2) is proved by replacing (2.1.7) with

$$(4.6.17) \quad (x_1 x_2 \cdots x_m) * (x_{m+1} * (x_{m+2} \cdots x_n)) \in (\text{Sym}_{\mathbb{S}}^m(\Sigma\mathfrak{g}))(\text{Sym}_{\mathbb{S}}^{n-m} \widehat{\mathfrak{g}}). \quad \square$$

4.7. Categorical generalization of the main results. — In fact, there is nothing in Theorem 4.5.1 that requires the category of \mathbb{S} -modules. Take any abelian symmetric monoidal category \mathcal{C} (see, e.g., [JS93]) with braiding β and arbitrary limits. Denote the monoidal product by \otimes : we will omit any subscripts of \mathcal{C} in this section. Take any Lie algebra $(\mathfrak{g}, \{, \})$ in the category \mathcal{C} : that is, an object in \mathcal{C} equipped with a morphism $\{, \} : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ which is skew-symmetric and satisfies the Jacobi identity (using β to express both).

One has symmetric and universal enveloping algebras $\text{Sym } \mathfrak{g}$ and $U\mathfrak{g}$ in the category \mathcal{C} , and a natural epimorphism

$$(4.7.1) \quad \text{Sym } \mathfrak{g} \twoheadrightarrow \text{gr } U\mathfrak{g}.$$

One similarly has the notion of pre-Lie algebras, and one can consider filtered associative star products $*$: $\text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g}$ in \mathcal{C} whose associated graded is the morphism (4.7.1). Moreover, $\text{Sym } \mathfrak{g}$ is naturally equipped with the structure of a bialgebra in \mathcal{C} : let I be the identity object of \mathcal{C} with respect to tensor product. The coproduct

$$(4.7.2) \quad \Delta : \text{Sym } \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g}$$

is then defined by uniquely extending the canonical morphism $\mathfrak{g} \rightarrow ((I \otimes \mathfrak{g}) \oplus (\mathfrak{g} \otimes I))$ so as to obtain a bialgebra. We will be interested in viewing $\text{Sym } \mathfrak{g}$ as a filtered coalgebra, and equipping it with a new associative structure $*$ whose associated graded is the original bialgebra structure on $\text{Sym } \mathfrak{g}$.

In this context, Theorem 4.5.1 goes through without change:

Theorem 4.7.3. — *Pre-Lie algebras in \mathcal{C} are equivalent to star-product algebra structures on $\text{Sym } \mathfrak{g}$ satisfying*

- (i) $*$ forms a bialgebra in \mathcal{C} with the usual coproduct Δ ,
- (ii) $*$ is a filtered product whose associated graded is the usual product on $\text{Sym } \mathfrak{g}$,
- (iii) $*$ satisfies

$$(4.7.4) \quad (\text{Sym}^m \mathfrak{g}) * \text{Sym } \mathfrak{g} \subseteq (\text{Sym}^m \mathfrak{g})(\text{Sym } \mathfrak{g}) = \text{Sym}^{\geq m} \mathfrak{g},$$

are equivalent to (right) pre-Lie algebra structures $\circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$, under the correspondence that the restriction of $*$ to a morphism of \mathcal{C} ,

$$(4.7.5) \quad * : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g},$$

is the direct sum of the morphisms

$$(4.7.6) \quad \cdot : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \text{Sym}^2 \mathfrak{g}, \quad \circ : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g} \cong \text{Sym}^1 \mathfrak{g},$$

the first coming from the standard algebra structure of $\text{Sym } \mathfrak{g}$, and the second from the pre-Lie structure on \mathfrak{g} .

Moreover, Proposition 1.2.2 has a straightforward categorical generalization as follows. Let μ and μ_* denote the usual and star-product multiplications $\text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g} \rightarrow \text{Sym } \mathfrak{g}$, and μ_\circ denote the pre-Lie multiplication on \mathfrak{g} , which we will extend formally to a binary operation on $\text{Sym } \mathfrak{g}$ (which is not pre-Lie). Let $\iota_r : X \xrightarrow{\sim} X \otimes I$ be the natural isomorphism which is part of the identity structure of \mathcal{C} (for all objects X in \mathcal{C}). Then, as in Proposition 1.2.2, the following formulas hold and give an inductive (on degree of $\text{Sym } \mathfrak{g}$) way to define μ_* (while simultaneously extending μ_\circ to a binary operation on all of $\text{Sym } \mathfrak{g}$):

- (0) $\mu_\circ \iota_r = \text{Id}$,
- (1) $\mu_\circ(\text{Id} \otimes \mu_\circ)|_{\text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g} \otimes \mathfrak{g}} = \mu_\circ(\mu_\circ \otimes \text{Id} - \text{Id} \otimes \mu_\circ)|_{\text{Sym } \mathfrak{g} \otimes \text{Sym } \mathfrak{g} \otimes \mathfrak{g}}$,
- (2) $\mu_\circ(\mu_\circ \otimes \text{Id}) = \mu_\circ(\mu_\circ \otimes \mu_\circ)(\text{Id} \otimes \beta \otimes \text{Id})(\text{Id} \otimes \text{Id} \otimes \Delta)$,
- (3) $\mu_* = \mu_\circ(\mu_\circ \otimes \text{Id})(\text{Id} \otimes \Delta)$.

Corollary 4.7.7. — *If \mathfrak{g} is a pre-Lie algebra (or associative algebra) in \mathcal{C} and $U\mathfrak{g}$ the universal enveloping algebra of the associated Lie algebra, then*

1. *The canonical morphism $\text{Sym } \mathfrak{g} \rightarrow \text{gr}(U\mathfrak{g})$ is an isomorphism;*
2. *The canonical morphism lifts to a coalgebra isomorphism $\text{Sym } \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$.*

Sketch of proof of Theorem 4.7.3. — The proof of this theorem is a straightforward generalization of the proof of Theorem 1.1.4 in the case when (4.7.1) is an isomorphism. In more detail, we translate the formulas there into categorical terms, analogously to the way we translated (4.5.3) into (4.7.5) and (4.7.6), and Proposition 1.2.2.(0)–(3) into the above.

To avoid assuming that (4.7.1) is an isomorphism, we can prove a suspended version of the above. This involves considering the category $\mathbb{S}_{\mathcal{C}}$, whose objects are of the form $\bigoplus_{i \geq 0} X_i$, where X_i are objects of \mathcal{C} equipped with an action by S_i of automorphisms. For example, if \mathcal{C} is the category of \mathbf{k} -modules, then $\mathbb{S}_{\mathcal{C}} = \mathbb{S}$ -modules, and if $\mathcal{C} = \mathbb{S}$ -modules, then $\mathbb{S}_{\mathcal{C}} = \mathbb{S}$ -bimodules (as remarked in §4.5). Analogously to the case of \mathbb{S} -modules, we can endow $\mathbb{S}_{\mathcal{C}}$ with the structure

of symmetric monoidal category, given by (3.2.2) and (3.2.3) verbatim, provided we understand the operation $\text{Ind}_H^G(X)$ for finite groups $H < G$, and X an object of \mathcal{C} equipped with an action of H by automorphisms. One way to construct this is to take $X^{\oplus |G|}$, labeling the copies of X by the elements of G , and then quotient by setting the diagonal action of H by automorphisms equal to the action of H by permuting the factors of X according to its action on G .

Then, we can form the object $\Sigma \mathfrak{g}[t]$ categorically. If \mathfrak{g} is an object of \mathcal{C} , we form the object $\Sigma \mathfrak{g} = \bigoplus_{i \geq 0} X_i$ where $X_1 = \mathfrak{g}$ and all other $X_i = 0$ (equipped with the trivial action of S_1). We can also form $\Sigma \mathfrak{g}[t] = \bigoplus_{i \geq 0} Y_i$ where $Y_i = \bigoplus_{j=1}^i I^{\otimes j-1} \otimes \mathfrak{g} \otimes I^{\otimes i-j} \cong \mathfrak{g}^{\oplus i}$, equipped with the action of S_i by permutation of components.

Now, $\Sigma \mathfrak{g}[t]$ is not merely an object of $\mathbb{S}_{\mathcal{C}}$, but a module over the algebra $P := \text{Sym}(\Sigma I)$, which replaces the polynomial algebra $\mathbf{k}[t]$ (ΣI replaces $\langle t \rangle$).

A straightforward categorical generalization of Proposition 3.3.3 then shows that pre-Lie structures on objects \mathfrak{g} of \mathcal{C} are equivalent to Lie structures on $\Sigma \mathfrak{g}[t]$ in $\mathbb{S}_{\mathcal{C}}$ compatible with the P -algebra structure (with P as in the preceding paragraph). The proof, as in the case of Proposition 3.3.3, is just a matter of checking the definitions, and expanding the Jacobi identity in $\mathbb{S}_{\mathcal{C}}$ into the components of

$$(4.7.8) \quad (\Sigma \mathfrak{g}[t])_3 = (\mathfrak{g} \otimes I \otimes I) \oplus (I \otimes \mathfrak{g} \otimes I) \oplus (I \otimes I \otimes \mathfrak{g}).$$

Now, as we will outline in Section 5, Stover's graded PBW theorem for connected Lie algebras extends to the categorical setting of $\mathbb{S}_{\mathcal{C}}$. This allows us to prove that $\text{Sym}_{\mathbb{S}_{\mathcal{C}}, P}(\Sigma \mathfrak{g}[t]) \xrightarrow{\sim} U_{\mathbb{S}_{\mathcal{C}}, P}(\Sigma \mathfrak{g}[t])$ via a star-product on the former symmetric algebra. Finally, we can take S_n -coinvariants in each degree n , and perform the identifications $I^{\otimes j-1} \otimes \mathfrak{g} \otimes I^{\otimes i-j} \cong \mathfrak{g}$ which generalize setting $t = 1$. We conclude that $\text{Sym}_{\mathcal{C}} \mathfrak{g} \xrightarrow{\sim} U_{\mathcal{C}} \mathfrak{g}$ via the star product $*$ asserted in the theorem. \square

Remark 4.7.9. — There is a common generalization of Theorems 4.7.3 and 4.6.10, which extends the suspension functor to act from $\mathbb{S}_{\mathcal{C}}$ to itself, taking pre-Lie algebras to pre-Lie algebras, and such that pre-Lie algebra structures on $\mathfrak{g} \in \mathbb{S}_{\mathcal{C}}$ are equivalent to certain star products on $\text{Sym}_{\mathbb{S}_{\mathcal{C}}, P} \Sigma \mathfrak{g}[t]$ (with notations as in the proof above). We omit the details.

5. Graded PBW theorems in a unified categorical context

In this section, we provide a simple proof of the twisted and non-twisted graded PBW theorems in a more general categorical context. We also generalize Stover's graded PBW theorem [Sto93] for connected twisted Lie algebras from the setting of \mathbf{k} -modules to that of an arbitrary symmetric monoidal category (Theorem 5.0.10.(iv)).

We will use the setup of §4.7. In this section, we address the question: when is (4.7.1) an isomorphism? To answer this, let $J \subset T\mathfrak{g}$ be the kernel of the projection $T\mathfrak{g} \twoheadrightarrow U\mathfrak{g}$. For all n , let $J_{\leq n}$ be the subobject of $T^{\leq n} \mathfrak{g}$ given by

$$(5.0.1) \quad J_2 := (\text{Id} - \beta - \{, \}) (T^2 \mathfrak{g}) \subset T^{\leq 2} \mathfrak{g},$$

$$(5.0.2) \quad J_{\leq n} := \sum_{i+j+2 \leq n} T^i \mathfrak{g} \otimes J_2 \otimes T^j \mathfrak{g}, \quad \forall n \geq 3.$$

Then, it is evident that $J = \lim_{n \rightarrow \infty} J_{\leq n}$.

The main technical result we need is

Lemma 5.0.3. — *The formula*

$$(5.0.4) \quad (i, i+1) \cdot \{, \} := \beta^{i, i+1} + \{, \}^{i, i+1}$$

defines an action of S_n on $T^n \mathfrak{g} \oplus (T^{\leq n-1} \mathfrak{g} / J_{\leq n-1} \mathfrak{g})$ for all n , acting trivially on the second factor, so that taking coinvariants yields $T^{\leq n} \mathfrak{g} / J_{\leq n} \mathfrak{g}$.

Proof. — We need to check the following identities:

$$(5.0.5) \quad (i, i+1)^2 \cdot_{\{\cdot, \cdot\}} = \text{Id},$$

$$(5.0.6) \quad (i, i+1)(i+1, i+2)(i, i+1) \cdot_{\{\cdot, \cdot\}} = (i+1, i+2)(i, i+1)(i+1, i+2) \cdot_{\{\cdot, \cdot\}},$$

$$(5.0.7) \quad (i, i+1)(j, j+1) \cdot_{\{\cdot, \cdot\}} = (j, j+1)(i, i+1) \cdot_{\{\cdot, \cdot\}}.$$

The first identity (5.0.5) follows from skew-symmetry of $\{\cdot, \cdot\}$. The braid relation (5.0.6) follows from the Jacobi identity. Finally, (5.0.7) follows by definition. \square

Now, there is a natural exact sequence of S_n -modules

$$(5.0.8) \quad 0 \rightarrow T^{\leq n-1} \mathfrak{g} / J_{\leq n-1} \mathfrak{g} \rightarrow (T^n \mathfrak{g} \oplus (T^{\leq n-1} \mathfrak{g} / J_{\leq n-1} \mathfrak{g}), \cdot_{\{\cdot, \cdot\}}) \rightarrow T^n \mathfrak{g} \rightarrow 0,$$

where, on the left, $T^{\leq n-1} \mathfrak{g} / J_{\leq n-1} \mathfrak{g}$ is given the trivial action, and on the right, $T^n \mathfrak{g}$ is given the standard S_n -action. We deduce

Proposition 5.0.9. — *The graded PBW map (4.7.1) is an isomorphism if and only if, for all n , the sequence (5.0.8) remains exact after taking S_n -coinvariants.*

For brevity, say that *the graded PBW theorem holds* for \mathfrak{g} if (4.7.1) is an isomorphism.

Theorem 5.0.10. — (i) *If \mathcal{C} is enriched over \mathbb{Q} -vector spaces, then the graded PBW theorem holds.*⁽⁶⁾

- (ii) *(Usual graded PBW theorem): If \mathcal{C} is the category of \mathbf{k} -modules, and \mathfrak{g} is a Lie algebra in \mathcal{C} which is free as a \mathbf{k} -module, then the graded PBW theorem holds if and only if $\{x, x\} = 0$ for all $x \in \mathfrak{g}$.*
- (iii) *(Twisted graded PBW theorem [Sto93]): If \mathcal{C} is the category of \mathbb{S} -modules, and \mathfrak{g} is a connected Lie algebra in \mathcal{C} , then the graded PBW theorem holds. More generally, if \mathfrak{g}_0 is a free \mathbf{k} -module, the graded PBW theorem holds if and only if $\{x, x\} = 0$ for all $x \in \mathfrak{g}_0$.*
- (iv) *(Categorical version of (iii)): If \mathcal{C} is arbitrary and $\mathbb{S}_{\mathcal{C}}$ is the category of symmetric sequences of objects of \mathcal{C} (cf. §4.7), and \mathfrak{g} is a connected Lie algebra in $\mathbb{S}_{\mathcal{C}}$, then the graded PBW theorem holds.*
- (v) *(Pre-Lie PBW theorem): For arbitrary \mathcal{C} , if \mathfrak{g} is the associated Lie algebra of a pre-Lie algebra in \mathcal{C} , then the graded PBW theorem holds.*

Note that (iii) is a generalization of (ii). Also, by a similar argument, one can replace “free” in (ii) and (iii) by the condition of being projective or a direct sum of cyclic modules.

Proof. — (i) If \mathcal{C} is enriched over \mathbb{Q} -vector spaces, then all S_n actions are actually $\mathbb{Q}[S_n]$ actions, and since $\mathbb{Q}[S_n]$ is semisimple, taking S_n -coinvariants is exact.

(ii) In this case, as an S_n -module, $T^n \mathfrak{g}$ is a direct sum of modules $\text{Ind}_{S_{p_1} \times \dots \times S_{p_r}}^{S_n} M$, where M is a free \mathbf{k} -module of rank one spanned by an element of the form $x_{i_1}^{\otimes p_1} \otimes \dots \otimes x_{i_r}^{\otimes p_r}$, for (x_i) a fixed \mathbf{k} -basis of \mathfrak{g} , and $i_1 \leq i_2 \leq \dots \leq i_r$. Thus, there is a splitting of the surjection in (5.0.8) if $\{x, x\} = 0$ for all x . Conversely, if $\{x, x\} \neq 0$ for some x , then the element $\{x, x\}$ is already in the kernel of $\mathfrak{g} \twoheadrightarrow \text{gr}_1 U \mathfrak{g}$.

(iii) By definition, for all n, m , $T^n(\mathfrak{g}_{>0})_m \cong \bigoplus_{m_1 + \dots + m_n = m} \text{Ind}_{S_{m_1} \times \dots \times S_{m_n}}^{S_m} (\mathfrak{g}_{m_1} \otimes \dots \otimes \mathfrak{g}_{m_n})$. Here, $m_i > 0$ for all i . Hence, S_n acts freely on the left cosets $S_m / (S_{m_1} \times \dots \times S_{m_n})$. Let $\mathcal{K}_{m,n}$

⁽⁶⁾This was noticed in [Fre98, Theorem A.9]; as explained there, it can also be proved in a usual manner.

be a set of representatives for these cosets. Thus, $T^n(\mathfrak{g}_{>0})_m$ is a direct sum of induced modules $\text{Ind}_{\{1\}}^{S_n}(\sigma(\mathfrak{g}_{m_1} \otimes \cdots \otimes \mathfrak{g}_{m_n}))$ for $\sigma \in \mathcal{K}_{m,n}$, and $m_1 \leq \cdots \leq m_n$. Hence, an S_n -module splitting of the surjection in (5.0.8) in degree m can be obtained from $S_{m_1} \times \cdots \times S_{m_n}$ -module splittings restricted to $\mathfrak{g}_{m_1} \times \cdots \times \mathfrak{g}_{m_n}$ for all m_1, \dots, m_n . This proves (iii) in the case that \mathfrak{g} is connected. The general case is a combination of this argument with that of (ii).

(iv) Just as in (iii), $T^n(\mathfrak{g}_{>0})_m$ is a direct sum of induced modules $\text{Ind}_{\{1\}}^{S_n}(\sigma(\mathfrak{g}_{m_1} \otimes \cdots \otimes \mathfrak{g}_{m_n}))$ for $\sigma \in \mathcal{K}_{m,n}$, and $m_1 \leq \cdots \leq m_n$, where induction is defined in §4.7. Hence, one obtains S_n -module splittings of the surjection of (5.0.8) in degree m as in (iii).

(v) This is Corollary 4.7.7.(i). \square

Remark 5.0.11. — In cases (i), (ii), and (v) of the theorem, in fact the quantum PBW theorem holds: one may lift the PBW isomorphism $\text{Sym } \mathfrak{g} \xrightarrow{\sim} \text{gr}(U\mathfrak{g})$ to a *coalgebra isomorphism* $\text{Sym } \mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$. This is *not* always true in cases (iii) and (iv), by Remark 4.1.5. However, note that, as in [Sto93] or the proof of (iii), one can at least lift (4.7.1) to an isomorphism of \mathbf{k} -modules, $\text{Sym}_{\mathbb{S}} \mathfrak{g} \xrightarrow{\sim} U_{\mathbb{S}} \mathfrak{g}$, and in the case of (iv), to an isomorphism in the category \mathcal{C} , $\text{Sym}_{\mathbb{S}_{\mathcal{C}}} \mathfrak{g} \xrightarrow{\sim} U_{\mathbb{S}_{\mathcal{C}}} \mathfrak{g}$ (just not necessarily in the category $\mathbb{S}_{\mathcal{C}}$).

Remark 5.0.12. — In other symmetric monoidal categories, one can always find extra conditions on the bracket $\{, \}$ so that the graded PBW theorem still holds. In general, the graded PBW theorem holds if and only if, whenever a sum of terms of the form

$$(5.0.13) \quad a \otimes (x \otimes y - \beta(x \otimes y)) \otimes b$$

is zero, for $x, y \in \mathfrak{g}$ and $a, b \in T\mathfrak{g}$, then also the corresponding sum of terms

$$(5.0.14) \quad a \otimes \{x, y\} \otimes b$$

is zero. For this to be valid for a symmetric monoidal category where elements of \mathfrak{g} don't exist, we replace the above by the condition that, for all $n \geq 2$, the kernel of the map $(\mathfrak{g}^{\otimes n})^{\oplus(n-1)} \rightarrow \mathfrak{g}^{\otimes n}$ given by $\bigoplus_i (\text{Id} - \beta)^{i, i+1}$ injects into the kernel of the map $(\mathfrak{g}^{\otimes n})^{\oplus(n-1)} \rightarrow \mathfrak{g}^{\otimes(n-1)}$ given by $\bigoplus_i \{, \}^{i, i+1}$. The case $n = 2$ of this is the condition that

$$(5.0.15) \quad \ker(\beta - \text{Id}) \subseteq \ker(\{, \}), \quad \text{in } \mathfrak{g}^{\otimes 2}.$$

This may be viewed as the generalized alternating condition. This is not enough in some cases, e.g., in the case of Lie superalgebras (see Remark 1.4.1).

Remark 5.0.16. — There are various generalizations of the graded PBW theorem to the case of quotients of $T\mathfrak{g}$ by ideals that resemble J above, such as quantized enveloping algebras: see, e.g., [Ber92]. These should also have categorical generalizations, which one should be able to prove by modifying the above approach.

A

PBW counterexamples and pre-Lie identities

Here, we recall the example of [Coh63, §5] where the graded PBW theorem does not hold, i.e., $\text{Sym } \mathfrak{g} \twoheadrightarrow \text{gr } U\mathfrak{g}$ is not injective, exploiting a classical p -th power identity of Zassenhaus. We remark that the pre-Lie graded PBW theorem (Corollary 1.3.1.(i)) implies that the identity must lift to the pre-Lie setting, which explains such an identity observed in [Tou06].⁽⁷⁾

⁽⁷⁾I am grateful to J.-L. Loday, whose question sparked this appendix.

Zassenhaus observed in [Zas39] that there exists a Lie polynomial $\Lambda_p(x, y)$ such that

$$(A.0.1) \quad (x + y)^p - x^p - y^p \equiv \Lambda_p(x, y) \pmod{p}.$$

In [Coh63], this identity was exploited to give an example where the PBW map $\text{Sym } \mathfrak{g} \rightarrow \text{gr } U\mathfrak{g}$ fails to be an isomorphism. Let \mathbb{F} be a field of characteristic $p > 0$, and let $\mathbf{k} = \mathbb{F}[\alpha, \beta, \gamma]/(\alpha^p, \beta^p, \gamma^p)$. Let L be the \mathbf{k} -module presented as $L = \text{Span}(x, y, z)/(\alpha x = \beta y + \gamma z)$. Let \mathfrak{g} be the free Lie algebra over \mathbf{k} generated by L .⁽⁸⁾

Then, it is evident that $\Lambda_p(\beta y, \gamma z) \neq 0$, since Λ is a Lie polynomial of degree p , and so the relations $\alpha^p = \beta^p = \gamma^p = 0$ cannot affect the expansion, as iterated brackets of p copies of the same element yields zero. On the other hand, (A.0.1) together with the fact that $U\mathfrak{g} = TL$ shows that $\Lambda_p(\beta y, \gamma z) = 0$ in $U\mathfrak{g}$, and hence also in $\text{gr } U\mathfrak{g}$. So $\mathfrak{g} \rightarrow U\mathfrak{g}$ is not injective, and neither is the canonical morphism $\text{Sym } \mathfrak{g} \rightarrow \text{gr } U\mathfrak{g}$.

Now, if we consider instead of \mathfrak{g} the free pre-Lie algebra generated by L , which we denote by $\tilde{\mathfrak{g}}$, it follows from the inclusion $\mathfrak{g} \rightarrow \tilde{\mathfrak{g}}$ that $\Lambda_p(\beta y, \gamma z) = 0$ in $U\tilde{\mathfrak{g}}$. Since the graded PBW theorem must hold for all pre-Lie algebras (Corollary 1.3.1.(i)), we deduce that $\Lambda_p(x, y)$ must be in the linear span of all compositions of x with itself p times, of y with itself p times, and of $x + y$ with itself p times. Indeed, as observed in [Tou06, (7-8)], one has

$$(A.0.2) \quad (x + y)^{\circ p} - x^{\circ p} - y^{\circ p} \equiv \Lambda_p(x, y) \pmod{p},$$

where $y \circ^i x := (\cdots ((y \circ x) \circ x) \cdots \circ x)$ is the i -th power of the right action of x on y by \circ , and $x^{\circ i} := x \circ^{i-1} x$.

Remark A.0.3. — The identity (A.0.2) together with the fact that \circ is a right Lie action (this is equivalent to the definition of a pre-Lie algebra) and the relation $\text{ad}(x^p) \equiv \text{ad}^p x \pmod{p}$ for associative algebras in characteristic p yields the identities (where $\text{ad}(x)y := [x, y]$):

$$(A.0.4) \quad (\text{ad}^p(x + y) - \text{ad}^p(x) - \text{ad}^p(y))z \equiv \text{ad}((x + y)^{\circ p} - x^{\circ p} - y^{\circ p})z \pmod{p},$$

$$(A.0.5) \quad z \circ^p (x + y) - z \circ^p x - z \circ^p y \equiv z \circ (x + y)^{\circ p} - z \circ x^{\circ p} - z \circ y^{\circ p} \pmod{p}.$$

However, as pointed out in [Tou06, §7] (and is easy to check), the operation $x \mapsto x^{\circ p}$ fails to yield a restricted Lie algebra structure, i.e., $\text{ad}^p(x)z \not\equiv \text{ad}(x^{\circ p})z \pmod{p}$, in general. Similarly, $z \circ^p x \not\equiv z \circ x^{\circ p} \pmod{p}$, in general.

This leads one to ask: is there an example of a restricted Lie algebra, or more generally, a Lie algebra \mathfrak{g} over a characteristic- p base ring \mathbf{k} with a p -th power operation satisfying (A.0.2), (A.0.4), and $(\alpha x)^{\circ p} = \alpha^p x^{\circ p}$ (for all $\alpha \in \mathbf{k}$ and $x \in \mathfrak{g}$), such that the (nonrestricted) PBW morphism $\text{Sym } \mathfrak{g} \rightarrow \text{gr } U\mathfrak{g}$ fails to be injective?

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⁽⁸⁾Note that \mathfrak{g} is in fact graded and connected.

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March 23, 2010

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